Solvability of Equations in Clifford Algebras

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Abstract

In this paper, we are studying selected types of quadratic equations in Clifford algebra, using methods developed for solving analogous equations in quaternions. Our goal is to classify the solutions in order to build a solid foundation for the study of Minkowski Pythagorean hodograph curves.

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Chapter 1

Introduction

In the 19$^{\text{th}}$ and early 20$^{\text{th}}$ century, two mathematical formalisms emerged and competed in the physics discourse, motivated by the vast amount of new discoveries made at the time (to name one, the Maxwell equations). One was the quaternions, a non-commutative generalization of complex numbers to 4 dimensions, first described but unpublished by Carl Friedrich Gauss in 1819, once again in 1840 without public recognition by Benjamin Olinde Rodrigues and famously by Sir William Rowan Hamilton in 1843 [1]. The other formalism was what we now know as vector calculus and analysis, derived from the quaternions theory independently by Josiah Willard Gibbs and Oliver Heaviside. As the latter proved to be the more pragmatic, it prevailed, while quaternions faded into obscurity, remaining merely a topic within the realm of pure mathematics, until the end of the 20$^{\text{th}}$ century, when their advantages over vector calculus found appreciation and applications in, among other fields, engineering and computer science [6, p. 63].

Both quaternions and vector calculus focus on the geometric transformations in 3-dimensional space (the quaternions' extension to four dimensions is to be viewed as a tool for preserving the algebraic properties); however, the foundation for generalization of multi-dimensional numbers to an arbitrary number of dimensions was already being laid at the time. The first steps towards this goal were, again without much public recognition, taken by German teacher Hermann Günther Graßmann in the 1830's. Graßmann single-handedly developed the idea of a vector space, some 80 years before Weyl, while also including the idea of an exterior product, an operation which yields oriented entities of higher dimensions. For instance, the exterior product of two ordinary vectors (which themselves may be regarded as oriented line segments) is an oriented parallelogram - a bivector [15, ch. 1]. In 1876, Graßmann's algebraic theory was unified with Hamilton's geometrically motivated concept of quaternions to a single system - the geometric or Clifford algebra - by Englishman William Kingdon Clifford. Much like quaternions, Clifford algebras had to wait until the second half of the 20$^{\text{th}}$ century to be rediscovered by physicists and find their place in applied mathematics, however, they are now being used in variety of subjects ranging from multiple areas in mathematical physics (both classical and relativistic mechanics and quantum theory) to robotics, computer graphics and computer-aided geometric design [15, 19, 22, 24, 17].

Our motivation for the study of Clifford algebras comes from the theory of Pythagorean-hodograph and Minkowski-Pythagorean-hodograph curves (abbreviated as PH and MPH curves, respectively). PH curves were introduced in $\mathbb{R}^2$ in 1990 and in $\mathbb{R}^3$ in 1994 by Rida T. Farouki and Takis Sakkalis [10, 11] as a type of polynomial curves satisfying the *Pythagorean property*:

$$x_1'^2(t) + x_2'^2(t) + \cdots + x_n'^2(t) = \sigma^2(t). \tag{1.0.1}$$

This property can be regarded as a weaker analogy to the arc-length parametrization and gives PH curves their excellent properties when used as interpolants. In his monograph on the topic [6], Farouki uses a neat and concise representation of planar and spatial PH curves using complex numbers and quaternions, respectively, with the left hand side of the Pythagorean condition (1.0.1) being a squared norm of so-called preimage curve, which resides in the complex or quaternion space.

Minkowski-Pythagorean-hodograph curves are PH curves residing in a Minkowski space - a real vector space equipped with an indefinite inner product (as opposed to the Euclidean space, which is equipped with a positive-definite inner product). This has a very specific motivation, the Medial Axis Transform or MAT, which is a structure describing areas or volumes as envelopes of families of circles or spheres of varying radii $r(t)$, whose centers lie on a common curve, the medial axis. In an equation for such an envelope in, for instance, the planar setting, we find the following term:

$$\sqrt{x'^2(t) + y'^2(t) - r'^2(t)}. \tag{1.0.2}$$

If we can make this expression equal to some polynomial $\sigma(t)$ (and the medial axis $(x(t), y(t))$ is also polynomial), the resulting MAT and all of its offsets are rational. This leads to the use of a particular Clifford algebra over a Minkowski space, in which the expression (1.0.2) becomes a squared pseudonorm of the preimage [6, ch. 24].

From this area stems the immediate motivation for the topic of this thesis. Recently, two studies of solutions to quadratic equations in quaternions were published [8, 13]. Our goal is to offer a similar discussion for selected equations in the setting of the specific Clifford algebra used to model the MPH curves. As this algebra can be viewed as a "skew" version of the algebra of quaternions, we draw analogies to the quaternion case wherever possible. We hope to offer a solid foundation upon which further research of MPH curves could be built.

The thesis is organized as follows:

- In Chapter 2 we introduce general vector and quadratic spaces, Minkowski spaces as a particular examples of quadratic spaces, quaternions and finite-dimensional Clifford algebras.

- In Chapter 3 we give an overview of the methods for solving left-sided quadratic equations over the quaternions, then we present our solution to this type of equation in even subalgebras of two kinds of Clifford algebras used in modeling MPH curves.

- In Chapter 4 we will investigate a special case of an equation with mixed coefficients in a four-dimensional Clifford algebra.

- In Chapter 5 we summarize our results.

Chapter 2

Definitions

2.1 Background from rings and fields

We will begin by stating a definition and two theorems concerning rings and fields.

Definition 2.1. Let R be a ring. The *characteristic* of R is the least positive integer n such that $\forall r \in R$:

$$nr = \underbrace{r + r + \cdots + r}_{n \text{ terms}} = 0. \tag{2.1.1}$$

If no such n exist, we say that the characteristic of R is 0. We denote the characteristic of R by $\operatorname{char} R$.

Theorem 2.1. [14, p. 259] *Every field $\mathbb{K}$ has either $\operatorname{char} \mathbb{K} = 0$ or $\operatorname{char} \mathbb{K} = p$ where p is a prime.* $\square$

Theorem 2.2 (Steinitz, 1910)**.** [14, p. 290] *Let $\mathbb{K}$ be a field. If $\operatorname{char} \mathbb{K} = p$ where p is prime, then $\mathbb{K}$ contains a subfield isomorphic to $\mathbb{Z}_p$. If $\operatorname{char} \mathbb{K} = 0$, then $\mathbb{K}$ contains a subfield isomorphic to $\mathbb{Q}$.* $\square$

2.2 Vector spaces and algebras

In this section, we will define elementary structures from the point of view of abstract algebra. Our main sources for these definitions are [14] and [2].

2.2.1 Vector spaces

Definition 2.2. Let $\mathbb{K}$ be a field with a unity denoted by $1_{\mathbb{K}}$, V be an abelian group under addition denoted by $+$, and ϕ be a map $\phi : \mathbb{K} \times \mathrm{V} \to \mathrm{V}$. The structure (V, ϕ) is called a *vector space over the field* $\mathbb{K}$ or *linear space* if the following conditions are satisfied $\forall \alpha, \beta \in \mathbb{K}$ and $\forall \mathbf{u}, \mathbf{v} \in \mathrm{V}$:

(V1) $$\phi(\alpha, \mathbf{u} + \mathbf{v}) = \phi(\alpha, \mathbf{u}) + \phi(\alpha, \mathbf{v}).$$

(V2) $$\phi(\alpha + \beta, \mathbf{u}) = \phi(\alpha, \mathbf{u}) + \phi(\beta, \mathbf{u}).$$

(V3) $$\phi(\alpha\beta, \mathbf{u}) = \phi(\alpha, \phi(\beta, \mathbf{u})).$$

(V4) $$\phi(1_{\mathbb{K}}, \mathbf{u}) = \mathbf{u}.$$

Elements of V are called *vectors*. The map ϕ is called *scalar multiplication* and shall from now on be denoted by juxtaposition; that is, $\phi(\alpha, \mathbf{u}) = \alpha\mathbf{u}$.

Example 2.1. The group of all n-tuples ($n \in \mathbb{N}$) of real numbers with element-wise addition is a vector space over the field $\mathbb{R}$ under scalar multiplication defined as:

$$\lambda(x_1, x_2, \ldots, x_n) = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n), \quad \lambda, x_1, x_2, \ldots, x_n \in \mathbb{R}. \tag{2.2.1}$$

We denote this vector space as $\mathbb{R}^n$.

Definition 2.3. Let V be a vector space over a field $\mathbb{K}$. A subgroup $\mathrm{U} \subset \mathrm{V}$ is called a *subspace* of the space V if it also is a vector space over $\mathbb{K}$.

Definition 2.4. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be vectors from V. The subspace

$$\langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \rangle := \left\{ \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n \,\middle|\, \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{K},\, n \in \mathbb{N} \right\} \tag{2.2.2}$$

is said to be *spanned* by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

Definition 2.5. The sum of vectors $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n$ as in equation (2.2.2) is called a *linear combination* of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

Definition 2.6. The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathrm{V}$ are said to be *linearly dependent* if there exists a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ such that

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n = 0, \tag{2.2.3}$$

where at least one of $\alpha_1, \alpha_2, \ldots, \alpha_n$ is non-zero. Vectors that are not linearly dependent are said to be *linearly independent*.

Definition 2.7. A linearly independent subset of $\mathrm{B} \subset \mathrm{V}$ is called a *basis of* V if V is spanned by B; in other words, every $\mathbf{v} \in \mathrm{V}$ is a linear combination of vectors from B. The coefficients in such a linear combination are called the *coordinates* of $\mathbf{v}$ with respect to B.

Example 2.2. The *standard* or *canonical basis* of the space $\mathbb{R}^n$ as defined in example 2.1 is formed by vectors $\mathbf{e}_1 = (1, 0, \ldots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \ldots, 0)$, $\ldots$, $\mathbf{e}_n = (0, 0, \ldots, 0, 1)$.

Theorem 2.3. [14, p. 354] *If* B_1 *and* B_2 *are bases of a vector space* V*, then* $|\mathrm{B}_1| = |\mathrm{B}_2|$*; in words, the numbers of elements in any base of a given vector space is invariant.* □

Definition 2.8. The number of elements in a basis $|\mathrm{B}|$ of a space V is called the *dimension of* V.

It is easy to show that any vector space over $\mathbb{R}$ of dimension n is isomorphic to $\mathbb{R}^n$ [2, p. 76].

2.2.2 Algebras

Definition 2.9. Let V be a vector space over a field $\mathbb{K}$ and θ be a map $\theta : \mathrm{V} \times \mathrm{V} \to \mathrm{V}$. The structure (V, θ) is called an *algebra over the field* $\mathbb{K}$ if θ is a *bilinear*; that is, $\forall \alpha \in \mathbb{K}$ and $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathrm{V}$:

(A1) $$\theta(\mathbf{u}, \mathbf{v} + \mathbf{w}) = \theta(\mathbf{u}, \mathbf{v}) + \theta(\mathbf{u}, \mathbf{w}).$$

(A2) $$\theta(\mathbf{u} + \mathbf{v}, \mathbf{w}) = \theta(\mathbf{u}, \mathbf{w}) + \theta(\mathbf{v}, \mathbf{w}).$$

(A3) $$\theta(\alpha\mathbf{u}, \mathbf{v}) = \theta(\mathbf{u}, \alpha\mathbf{v}) = \alpha\theta(\mathbf{u}, \mathbf{v}).$$

Furthermore, if

(A4) $$\theta(\mathbf{u}, \theta(\mathbf{v}, \mathbf{w})) = \theta(\theta(\mathbf{u}, \mathbf{v}), \mathbf{w}),$$

then (V, θ) is called an *associative algebra.* The map θ is called *algebraic product* or simply *multiplication.* As seen from properties (V3), (A3) and (A4), for associative algebras we can denote the multiplication by juxtaposition as well $\mathbf{u}(\mathbf{v}\mathbf{w}) = (\mathbf{u}\mathbf{v})\mathbf{w}$.

Example 2.3. Consider the vector space of all 2×2 matrices with real entries $\mathbb{R}^{2\times 2}$ and let $A = \left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} b_1 & b_2 \\ b_3 & b_4 \end{smallmatrix}\right)$ be any two matrices from $\mathbb{R}^{2\times 2}$. If we define the algebraic product as

$$AB := \begin{pmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{pmatrix}, \tag{2.2.4}$$

we obtain the familiar matrix algebra over $\mathbb{R}^{2\times 2}$.

Example 2.4. Consider the vector space $\mathbb{R}^2$ (see Example 2.1) and let $\mathbf{z}_1 = (x_1, y_1)$, $\mathbf{z}_2 = (x_2, y_2)$ be any two vectors from $\mathbb{R}^2$. If we define the algebraic product as

$$\mathbf{z}_1\mathbf{z}_2 := (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1), \tag{2.2.5}$$

we obtain the familiar algebra of *complex numbers* $\mathbb{C}$ over $\mathbb{R}$.

Example 2.5. Consider the vector space $\mathbb{R}^4$ and let $\mathcal{A} = (a, a_x, a_y, a_z)$, $\mathcal{B} = (b, b_x, b_y, b_z)$ be any two vectors from $\mathbb{R}^4$. If we define the algebraic product as

$$\begin{aligned} \mathcal{AB} := \big(&ab - a_xb_x - a_yb_y - a_zb_z, \\ &ab_x + ba_x + a_yb_z - a_zb_y, \\ &ab_y + ba_y + a_zb_x - a_xb_z, \\ &ab_z + ba_z + a_xb_y - a_yb_x\big), \end{aligned} \tag{2.2.6}$$

we obtain the algebra of *quaternions* $\mathbb{H}$, which we will explore further in Section 2.4.

2.3 Quadratic spaces

The theory of quadratic forms and quadratic spaces is motivated by the need to equip vector spaces with deeper concepts, mainly notions of metric and angle. Our main sources for this section are [2] and [4].

Definition 2.10. Let V be a vector space over a field $\mathbb{K}$. A map $b : \mathrm{V} \times \mathrm{V} \to \mathbb{K}$ is called a *bilinear form* if $\forall \alpha, \beta \in \mathbb{K}$ and $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathrm{V}$:

(B1) $$b(\alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w}) = \alpha b(\mathbf{u}, \mathbf{w}) + \beta b(\mathbf{v}, \mathbf{w}).$$

(B2) $$b(\mathbf{u}, \alpha\mathbf{v} + \beta\mathbf{w}) = \alpha b(\mathbf{u}, \mathbf{v}) + \beta b(\mathbf{u}, \mathbf{w}).$$

Furthermore, if

(B3) $$b(\mathbf{u}, \mathbf{v}) = b(\mathbf{v}, \mathbf{u}),$$

then b is called a *symmetric bilinear form*.

Example 2.6. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ be vectors from $\mathbb{R}^n$. The standard Euclidean inner product defined as

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

is a symmetric bilinear form.

Definition 2.11. Let V be a vector space over a field $\mathbb{K}$. A map $q : \mathrm{V} \to \mathbb{K}$ is called a *quadratic form* if $\forall \alpha \in \mathbb{K}$ and $\forall \mathbf{u}, \mathbf{v} \in \mathrm{V}$:

(Q1) $$q(\alpha \mathbf{u}) = \alpha^2 q(\mathbf{u}).$$

(Q2) $$q(\mathbf{u} + \mathbf{v}) - q(\mathbf{u}) - q(\mathbf{v}) = b_q(\mathbf{u}, \mathbf{v}),$$

where b_q is a symmetric bilinear form called the *associated bilinear form of* q. The structure (V, q) is called a *quadratic* or *inner product space.*

Example 2.7. The standard Euclidean inner product of a vector from $\mathbb{R}^n$ with itself

$$\mathbf{x} \cdot \mathbf{x} = x_1^2 + x_2^2 + \cdots + x_n^2$$

is a quadratic form.

Convention 2.1. From this point onward, all fields in this section are fields with characteristic not 2, as our goal is to study algebras over the field $\mathbb{R}$ which has $\operatorname{char} \mathbb{R} = 0$.

Theorem 2.4. *Every quadratic form q has its value fully determined by some symmetric bilinear form b and vice versa, through the identities:*

$$q(\mathbf{u}) := b(\mathbf{u}, \mathbf{u}). \tag{2.3.1a}$$

$$b(\mathbf{u}, \mathbf{v}) := \frac{1}{2}\big[q(\mathbf{u} + \mathbf{v}) - q(\mathbf{u}) - q(\mathbf{v})\big] = \frac{1}{2} b_q(\mathbf{u}, \mathbf{v}). \tag{2.3.1b}$$

Equation (2.3.1a) *is called the* polarization identity*. The bilinear form $b(\mathbf{u}, \mathbf{v})$ is called the* inner product *and will be denoted by the dot $(\cdot)$ operator $b(\mathbf{u}, \mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$.*

Proof. Equation (2.3.1a) is verified by observing that the right hand side satisfies properties (Q1) and (Q2), note that here, the associated bilinear form $b_q(\mathbf{u}, \mathbf{v}) = 2b(\mathbf{u}, \mathbf{v})$. Equation (2.3.1b) is obvious as multiplying by a constant preserves linearity. □

Remark 2.1. In the previous theorem, as we are working over a general field $\mathbb{K}$ of characteristic not 2, we use $\frac{1}{2}$ to denote the multiplicative inverse of 2 in the sense of Steinitz's Theorem 2.2.

Thanks to the correspondence established in Theorem 2.4, we can investigate properties of the inner product solely through the quadratic form associated with it. For the next theorem, we shall now limit ourselves to vector spaces of the type $\mathbb{R}^n$.

Theorem 2.5 (Sylvester's law of inertia)**.** *For any quadratic form q of $\mathbb{R}^n$, there exist numbers $p, r \in \mathbb{N}_0$, $p + r \leq n$, and a basis* B *of $\mathbb{R}^n$ such that for any vector $\mathbf{x} \in \mathbb{R}^n$ with coefficients $x_1, x_2, \ldots, x_n$ in the basis B:*

$$\mathbf{x} \cdot \mathbf{x} = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+r}^2. \tag{2.3.2}$$

Proof. The proof is performed by establishing correspondence between symmetric bilinear forms and symmetric matrices. See [2, p. 217-220] $\square$

The number $p+r$ is called the *rank of a quadratic form*, the ordered triple of numbers $(p, r, n-p-r)$ is called the *signature of a quadratic form*. The signature of a form is invariant regardless of the choice of basis satisfying the claim. The quadratic space equipped with such a quadratic form is denoted $\mathbb{R}^{p,r,n-p-r}$, $\mathbb{R}^{p,r}$ if $p + r = n$ or simply $\mathbb{R}^n$ if $p = n$.

Definition 2.12. Two distinct vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{p,r}$ are called *orthogonal* if $\mathbf{x} \cdot \mathbf{y} = 0$.

Definition 2.13. A *Minkowski space* is a vector space $\mathbb{R}^n$ equipped with a quadratic form with signature $(n-1, 1, 0)$. We denote this space $\mathbb{R}^{n-1,1}$. The inner product on this space is called *Minkowski inner product*. Based on the sign of Minkowski inner product of a vector with itself, we will classify vectors as follows:

- If $\mathbf{x} \cdot \mathbf{x} > 0$, $\mathbf{x}$ is called a *time-like* vector.
- If $\mathbf{x} \cdot \mathbf{x} < 0$, $\mathbf{x}$ is called a *space-like* vector.
- If $\mathbf{x} \cdot \mathbf{x} = 0$, $\mathbf{x}$ is called an *isotropic* or *light-like* vector.

The classification of vectors in Minkowski spaces originates from the theory of relativity, where it describes causal relations between events in spacetime, which is described as Minkowski space $\mathbb{R}^{3,1}$ or $\mathbb{R}^{1,3}$, as the signature varies in literature.

2.4 Quaternions

We shall now elaborate on the algebra of quaternions, which we introduced as an example of an algebra over the field $\mathbb{R}$ in Example 2.5. The motivation for creating such an algebra was the search for a system capable of describing spatial rotations analogously to description of planar rotations using complex numbers while preserving properties such as associativity, distributivity and commutativity of the algebraic product. It can easily be shown that no such algebra can have three dimensions [6, p. 64], however, a four dimensional algebra can be constructed if we do not require commutativity. As it turns out, the lack of commutativity has a geometrical foundation since spatial rotations do not generally commute.

2.4.1 Introduction to quaternions

Definition 2.14. A *quaternion* is an ordered 4-tuple of real numbers:

$$\mathcal{A} = [a, a_x, a_y, a_z].$$

The number a is the *scalar component* of a quaternion, the ordered triple $[a_x, a_y, a_z]$ is the *vector component* of a quaternion. A quaternion of the form $[a, 0, 0, 0]$ is called *pure scalar*, $[0, a_x, a_y, a_z]$ is called *pure quaternion*, $[0, 0, 0, 0]$ is called *zero quaternion.* Two quaternions are equal if all of their components are equal. The set of all quaternions is denoted by $\mathbb{H}$.[1] The sum of two quaternions is performed element-wise, the product of two quaternions is shown in Equation (2.2.6).

Remark 2.2. The quaternion product is generally non-commutative, however, scalar multiplication in the sense of a vector space operation is commutative and equivalent to multiplication by pure scalar quaternion, as one can easily verify from (2.2.6).

The following three propositions provide alternative ways that are used to study the quaternions.

Proposition 2.6. *Any quaternion $\mathcal{A} = (a, a_x, a_y, a_z)$ can be written in the* algebraic form *as*

$$\mathcal{A} = 1a + a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k},$$

[1] In honor of Sir William Rowan Hamilton.

where $1 = [1, 0, 0, 0]$, $\mathbf{i} = [0, 1, 0, 0]$, $\mathbf{j} = [0, 0, 1, 0]$ *and* $\mathbf{k} = [0, 0, 0, 1]$.

Proof. We need to prove that the definitions for sum and product of two quaternions hold. The proof for sum is obvious; for the product, we can from (2.2.6) establish the identities for quaternion units

$$\mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \quad \mathbf{ik} = -\mathbf{j} = -\mathbf{ki}, \quad \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \quad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1,$$

from which we can prove that the definition of product holds as well. □

The following two forms we present without verifying the consistency of sum and product.

Proposition 2.7. *Any quaternion* $\mathcal{A} = (a, a_x, a_y, a_z)$ *can be written in the* scalar-vector form *as*

$$\mathcal{A} = [a, \mathbf{a}],$$

where $\mathbf{a} \in \mathbb{R}^3$ *and the product is written as*

$$\mathcal{AB} = [ab - \mathbf{a} \cdot \mathbf{b}, a\mathbf{b} + b\mathbf{a} + \mathbf{a} \times \mathbf{b}],$$

with $\cdot$ *and* $\times$ *being standard dot and cross products in* $\mathbb{R}^3$. □

Proposition 2.8. *Any quaternion* $\mathcal{A} = (a, a_x, a_y, a_z)$ *can be written in the* matrix form *as*

$$\mathcal{A} = \begin{bmatrix} a & -a_x & -a_y & -a_z \\ a_x & a & -a_z & a_y \\ a_y & a_z & a & -a_x \\ a_z & -a_y & a_x & a \end{bmatrix},$$

where the multiplication is performed as standard matrix multiplication. □

Theorem 2.9. *The elements* 1, $\mathbf{i}$, $\mathbf{j}$ *and* $\mathbf{k}$ *form a basis of* $\mathbb{H}$.

Proof. It is easy to see that the elements are linearly independent and span the entire space $\mathbb{H}$. □

Definition 2.15. A *quaternion conjugate to* $\mathcal{A} = a + a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ is the quaternion

$$\mathcal{A}^* = a - a_x\mathbf{i} - a_y\mathbf{j} - a_z\mathbf{k}.$$

Definition 2.16. The *magnitude of* $\mathcal{A}$ is a positive pure scalar $|\mathcal{A}|$ which satisfies

$$|\mathcal{A}|^2 = \mathcal{A}\mathcal{A}^* = \mathcal{A}^*\mathcal{A}.$$

A quaternion with $|\mathcal{A}| = 1$ is called *unit quaternion.*

Definition 2.17. For any non-zero quaternion $\mathcal{A}$ we define the *inverse quaternion* $\mathcal{A}^{-1}$ as:

$$\mathcal{A}^{-1} = \frac{\mathcal{A}^*}{|\mathcal{A}|^2}.$$

2.4.2 Description of spatial rotations

Theorem 2.10. *The set of all unit quaternions forms a group under the quaternion product.*

Proof. We only need to show that the quaternion product preserves the magnitude of unit quaternions. This can be done the easiest using the matrix form. By direct computation, we can show that if $\mathcal{A}$ is in the matrix form, then

$$\det \mathcal{A} = |\mathcal{A}|^4, \tag{2.4.1}$$

therefore the unit magnitude is preserved under multiplication. □

Theorem 2.11. *Let $\mathcal{A}$ be a pure quaternion, then for any $\mathcal{B} \in \mathbb{H}$, $\mathcal{B}\mathcal{A}\mathcal{B}^*$ is also a pure quaternion.*

Proof. By direct computation. □

If we now examine the formula for the magnitude of a quaternion in the scalar-vector form, we can show that

$$|\mathcal{A}|^2 = a^2 + |\mathbf{a}|^2, \tag{2.4.2}$$

where $|\mathbf{a}|$ is the standard magnitude of a vector in $\mathbb{R}^3$. From this, we can deduce that any unit quaternion in the scalar-vector form can be written as

$$\mathcal{A} = \left[\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\mathbf{a}\right], \quad |\mathbf{a}| = 1, \tag{2.4.3}$$

for some angle $\theta \in [0, 2\pi)$. Consider now a vector $\mathbf{v} \in \mathbb{R}^3$ represented by a quaternion $\mathcal{V} = [0, \mathbf{v}]$ and a unit quaternion $\mathcal{U}$ of the form given in (2.4.3), representing a unit axis of rotation $\mathbf{u}$ and an angle θ. We can decompose $\mathbf{v}$ into components parallel and perpendicular to $\mathbf{u}$:

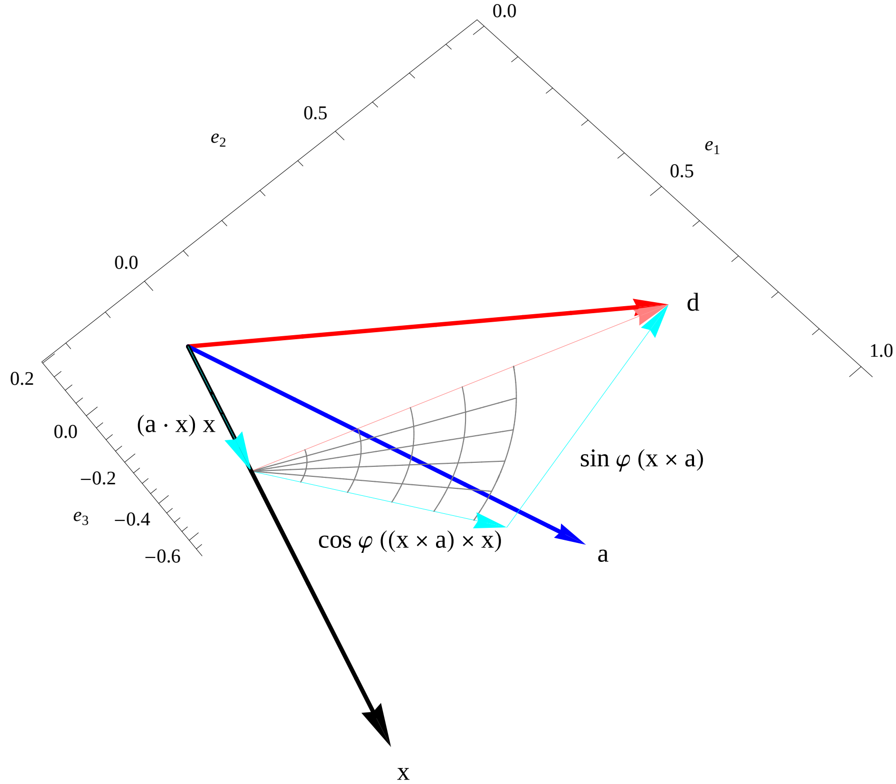
$$\mathbf{v} = (\mathbf{u} \cdot \mathbf{v})\mathbf{u} + (\mathbf{u} \times \mathbf{v}) \times \mathbf{u}. \tag{2.4.4}$$

Performing the product $\mathcal{U}\mathcal{V}\mathcal{U}^*$ yields

$$\mathcal{U}\mathcal{V}\mathcal{U}^* = [0, (\mathbf{u} \cdot \mathbf{v})\mathbf{u} + \sin\theta\,(\mathbf{u} \times \mathbf{v}) + \cos\theta\,((\mathbf{u} \times \mathbf{v}) \times \mathbf{u})]\,. \tag{2.4.5}$$

Observe that the component of $\mathbf{v}$ parallel to $\mathbf{u}$ remains unchanged while the component perpendicular to $\mathbf{u}$ underwent a rotation by angle θ, as illustrated in the Figure 2.1.

Figure 2.1: A visualization of rotation using quaternions.



2.5 Clifford algebras

A Clifford (or geometric) algebra can be viewed as a further extension of the concept multidimensional numbers, such as $\mathbb{C}$ or $\mathbb{H}$, to an arbitrary number of dimensions. The terms "Clifford" and "geometric" are often interchangeable, with the latter usually being used to emphasize an algebra constructed over a Euclidean space (this is also the term W. K. Clifford originally used). Therefore, as our area of interest is an algebra over a Minkowski space, we shall use the term "Clifford" throughout this thesis.

There are multiple ways of defining Clifford algebras, based on different motivations [19, ch. 14]. In this thesis, we will use a deductive approach, presented in [22]. This approach focuses on Clifford algebras of finite dimensions; however, in its most general form, as shown in [16], we can construct an infinite-dimensional Clifford algebra, of which all finite-dimensional Clifford algebras are subalgebras. Such approach is then fully independent on the choice of basis.

2.5.1 Finite-dimensional Clifford algebras

Definition 2.18. Let $\mathbb{R}^{2^{p+r}}$ be a vector space of dimension 2^{p+r} and let $\mathbb{R}$ and $\mathbb{R}^{p,r}$ be identified with distinct subspaces of $\mathbb{R}^{2^{p+r}}$. The *Clifford algebra* $\mathcal{C}\ell_{p,r}$ is an algebra over $\mathbb{R}^{2^{p+r}}$ with the algebraic product called the *Clifford* or *geometric product* (denoted by juxtaposition) satisfying

$$\mathbf{x}\mathbf{x} = \mathbf{x} \cdot \mathbf{x} \tag{C1}$$

for every $\mathbf{x} \in \mathbb{R}^{p,r}$, with the dot operator denoting an inner product on $\mathbb{R}^{p,r}$ with signature $(p, r, 0)$. The elements of $\mathcal{C}\ell_{p,r}$ are called *multivectors*.

The defining property of the algebraic product is that it maps the subspace of $\mathbb{R}^{2^{p+r}}$ corresponding to $\mathbb{R}^{p,r}$ to the subspace corresponding to $\mathbb{R}$. The geometric product is non-commutative, however we will identify algebraic multiplication by an element from the subspace $\mathbb{R}$ with scalar multiplication, which is commutative.

Convention 2.2. A power of any multivector (and thus consequently any vector) from $\mathcal{C}\ell_{p,r}$ is meant as a repeated Clifford product $\mathbf{x}^2 = \mathbf{x}\mathbf{x}$. Furthermore, the product operator $\prod$ is meant as a Clifford product $\prod_{i=1}^{2} \mathbf{x}_i = \mathbf{x}_1\mathbf{x}_2$.

Definition 2.19. For orthogonal vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ from $\mathbb{R}^{p,r} \subset \mathcal{C}\ell_{p,r}$, the Clifford product $\prod_{i=1}^{k} \mathbf{x}_i$ is called a *k-blade*.

With this definition, we may introduce the second axiom for Clifford algebra:

(C2) Every multivector can be written as a sum of k-blades.

Let $\mathbf{x}, \mathbf{y} \in \mathcal{C}\ell_{p,r}$ be any two vectors and consider the product

$$(\mathbf{x}+\mathbf{y})^2 = (\mathbf{x}+\mathbf{y}) \cdot (\mathbf{x}+\mathbf{y}) \tag{2.5.1}$$

If we expand the left hand side of (2.5.1) using the distributivity of an algebraic product ((A1) and (A2)) and the right hand side using distributivity and symmetry of an inner product ((B1), (B2) and (B3)), we obtain

$$\mathbf{x}^2 + \mathbf{xy} + \mathbf{yx} + \mathbf{y}^2 = \mathbf{x} \cdot \mathbf{x} + 2(\mathbf{x} \cdot \mathbf{y}) + \mathbf{y} \cdot \mathbf{y},$$

which after cancellation yields

$$\mathbf{x} \cdot \mathbf{y} = \frac{1}{2}(\mathbf{xy} + \mathbf{yx}). \tag{2.5.2}$$

Let us now denote

$$\mathbf{x} \wedge \mathbf{y} := \frac{1}{2}(\mathbf{xy} - \mathbf{yx}). \tag{2.5.3}$$

We can immediately see that

$$\mathbf{x} \wedge \mathbf{y} = -\mathbf{y} \wedge \mathbf{x}. \tag{2.5.4}$$

We can now write

$$\mathbf{xy} = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \wedge \mathbf{y}, \tag{2.5.5}$$

and if $\mathbf{x} \cdot \mathbf{y} = 0$, we get from (2.5.4)

$$\mathbf{xy} = \mathbf{x} \wedge \mathbf{y} = -\mathbf{y} \wedge \mathbf{x} = -\mathbf{yx}. \tag{2.5.6}$$

Lemma 2.12. *The expression $\mathbf{x} \wedge \mathbf{y}$ contains neither scalar nor vector part (in other words, the components corresponding to subspaces $\mathbb{R}$ and $\mathbb{R}^{p,r}$ are zero).*

Proof. The proof is taken from [5, p. 11]. Consider the expression $\mathbf{x}(\mathbf{x}\wedge\mathbf{y})$. From (2.5.3) we have

$$\begin{aligned}
\mathbf{x}(\mathbf{x}\wedge\mathbf{y}) &= \frac{1}{2}(\mathbf{x}^2\mathbf{y} - \mathbf{x}\mathbf{y}\mathbf{x}) \\
&= \frac{1}{2}(\mathbf{y}\mathbf{x}^2 - \mathbf{x}\mathbf{y}\mathbf{x}) \\
&= \frac{1}{2}(\mathbf{y}\mathbf{x} - \mathbf{x}\mathbf{y})\mathbf{x} \\
&= -(\mathbf{x}\wedge\mathbf{y})\mathbf{x}.
\end{aligned}$$

Therefore $\mathbf{x}\wedge\mathbf{y}$ contains no scalar part, since if it did, the product would contain a commutative part.

Next, assume $\mathbf{x}\wedge\mathbf{y}$ contains a vector part and denote it $\mathbf{z}$, then $\mathbf{z}(\mathbf{x}\wedge\mathbf{y}) + (\mathbf{x}\wedge\mathbf{y})\mathbf{z}$ must contain a scalar part, however, for any $\mathbf{w}$ satisfying $\mathbf{w}\cdot\mathbf{x} = \mathbf{w}\cdot\mathbf{y} = \mathbf{w}\cdot\mathbf{z} = 0$, we get

$$\begin{aligned}
\mathbf{w}\big(\mathbf{z}(\mathbf{x}\wedge\mathbf{y}) + (\mathbf{x}\wedge\mathbf{y})\mathbf{z}\big) &= \mathbf{w}\mathbf{z}(\mathbf{x}\wedge\mathbf{y}) + \mathbf{w}(\mathbf{x}\wedge\mathbf{y})\mathbf{z} \\
&= -\mathbf{z}\mathbf{w}(\mathbf{x}\wedge\mathbf{y}) + (\mathbf{x}\wedge\mathbf{y})\mathbf{w}\mathbf{z} \\
&= -\mathbf{z}(\mathbf{x}\wedge\mathbf{y})\mathbf{w} - (\mathbf{x}\wedge\mathbf{y})\mathbf{z}\mathbf{w} \\
&= -\big(\mathbf{z}(\mathbf{x}\wedge\mathbf{y}) + (\mathbf{x}\wedge\mathbf{y})\mathbf{z}\big)\mathbf{w},
\end{aligned}$$

since by (2.5.6)

$$\mathbf{w}(\mathbf{x}\wedge\mathbf{y}) = \frac{1}{2}(\mathbf{w}\mathbf{x}\mathbf{y} - \mathbf{w}\mathbf{y}\mathbf{x}) = \frac{1}{2}(-\mathbf{x}\mathbf{w}\mathbf{y} + \mathbf{y}\mathbf{w}\mathbf{x}) = \frac{1}{2}(\mathbf{x}\mathbf{y}\mathbf{w} - \mathbf{y}\mathbf{x}\mathbf{w}) = (\mathbf{x}\wedge\mathbf{y})\mathbf{w}.$$

Therefore, $\mathbf{z}(\mathbf{x}\wedge\mathbf{y}) + (\mathbf{x}\wedge\mathbf{y})\mathbf{z}$ contains no scalar part and thus $\mathbf{x}\wedge\mathbf{y}$ contains no vector part, which concludes the proof. □

We call the quantity $\mathbf{x}\wedge\mathbf{y}$ a *bivector*. A bivector can be visualized as an oriented parallelogram with sides corresponding to $\mathbf{x}$ and $\mathbf{y}$.

Definition 2.20. The wedge operator introduced in Equation (2.5.3) is called the *outer product*.

As a consequence of properties of the Clifford product, the outer product is associative and distributive.

2.5.2 Basis blades

Let $\overline{\mathbb{R}^{p,r}}$ denote the canonical basis of $\mathbb{R}^{p,r} \subset \mathcal{C}\ell_{p,r}$, $p + r = n$ as introduced in Example 2.2, and let the dot operator denote an inner product with signature $(p, r, 0)$. Vectors from this basis satisfy

$$\begin{aligned} \mathbf{e}_i \cdot \mathbf{e}_j &\neq 0 \quad \text{if } i = j, \\ \mathbf{e}_i \cdot \mathbf{e}_j &= 0 \quad \text{if } i \neq j, \end{aligned}$$

therefore from (2.5.6) we have

$$\mathbf{e}_i \mathbf{e}_j = -\mathbf{e}_j \mathbf{e}_i \quad \text{if } i \neq j.$$

Definition 2.21. Let $\overline{\mathbb{R}^{p,r}}[j]$ denote the j-th vector from $\overline{\mathbb{R}^{p,r}}$. Let the index set $\mathbb{I}$ be any subset of $\{1, 2, \ldots, n\}$ and let $\mathbb{I}[i]$ denote the i-th element of this set. A *basis blade* of $\mathcal{C}\ell_{p,r}$ is a Clifford product of vectors from $\overline{\mathbb{R}^{p,r}}$ of the form

$$\mathbf{e}_{\mathbb{I}} = \prod_{i}^{|\mathbb{I}|} \overline{\mathbb{R}^{p,r}}\big[\mathbb{I}[i]\big]. \tag{2.5.7}$$

The element corresponding to $\mathbb{I} = \emptyset$ is the scalar 1.

Example 2.8. If $\mathbb{I} = \{2, 3, 1\}$, then the basis blade corresponding to $\mathbb{I}$ is $\mathbf{e}_{231} = \mathbf{e}_2 \mathbf{e}_3 \mathbf{e}_1$.

As basis vectors from $\overline{\mathbb{R}^{k,m}}$ anticommute, we can reduce the number of basis blades as follows.

Definition 2.22. A *canonical basis blade* is a basis blade with index set $\mathbb{I}$ in ascending order.

Example 2.9. The basis blade from Example 2.8 can be rewritten as a canonical basis blade by two exchanges of neighboring vectors:

$$\mathbf{e}_{231} = \mathbf{e}_2 \mathbf{e}_3 \mathbf{e}_1 = -\mathbf{e}_2 \mathbf{e}_1 \mathbf{e}_3 = \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 = \mathbf{e}_{123}.$$

Theorem 2.13. *The set of all canonical basis blades forms a basis for* $\mathcal{C}\ell_{p,r}$.

Proof. The canonical basis blades are obviously linearly independent and any multivector can be expressed as some linear combination of them as a consequence of Axiom

(C2). Since there are 2^{p+r} ways to form a product of vectors as in Equation (2.5.7) to obtain a canonical basis blade, the dimension of $\mathcal{C}\ell_{p,r}$ is indeed $2^{p+r} = 2^n$ as was defined. $\square$

Example 2.10. The canonical basis for $\mathcal{C}\ell_3$ is:

$$\overline{\mathcal{C}\ell_3} = \{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{23}, \mathbf{e}_{123}\}.$$

Definition 2.23. Let $\mathbf{e}_{\mathbb{I}}$ be a basis blade. The *grade* of a basis blade is the number $|\mathbb{I}|$, i.e. the number of elements in the index set $\mathbb{I}$.

Definition 2.24. The *grade projection onto the grade k* operator is defined for basis blades as

$$\langle \mathbf{e}_{\mathbb{I}} \rangle_k := \begin{cases} \mathbf{e}_{\mathbb{I}} & \text{if } |\mathbb{I}| = k, \\ 0 & \text{otherwise,} \end{cases}$$

and is distributive over addition. The grade projection onto grade 0, i.e. the scalar part of a multivector, will be written without the subscript ($\langle \mathbf{e}_{\mathbb{I}} \rangle_0 = \langle \mathbf{e}_{\mathbb{I}} \rangle$).

Definition 2.25. A multivector $A \in \mathcal{C}\ell_{p,r}$ is said to be *homogeneous multivector of grade k* or *k-vector* if $\langle A \rangle_k = A$. A multivector A is said to be

- a (*pure*) *scalar* if $k = 0$,
- a (*pure*) *vector* if $k = 1$,
- a (*pure*) *bivector* if $k = 2$,
- a (*pure*) *trivector* if $k = 3$.

Using the notion of grade, we can now extend definitions of inner and outer products for pure vectors (2.5.2) and (2.5.3) to arbitrary multivectors.

Definition 2.26. Let A_k and B_m be homogeneous k-vector and m-vector respectively. Then their inner product is defined as

$$A_k \cdot B_m := \begin{cases} \langle A_k B_m \rangle_{|k-m|} & \text{if } k, m > 0, \\ 0 & \text{if } k = 0 \vee m = 0, \end{cases} \tag{2.5.8}$$

and for arbitrary multivectors A and B with maximum grades k and m respectively as

$$A \cdot B := \sum_{i=0}^{k} \sum_{j=0}^{m} \langle A \rangle_i \cdot \langle B \rangle_j. \tag{2.5.9}$$

Note that the inner product is a grade-lowering operation.

Definition 2.27. Let A_k and B_m be homogeneous k-vector and m-vector respectively. Their outer product is defined as

$$A_k \wedge B_m := \langle A_k B_m \rangle_{k+m}, \tag{2.5.10}$$

and for arbitrary multivectors A and B with maximum grades k and m respectively as

$$A \wedge B := \sum_{i=0}^{k} \sum_{j=0}^{m} \langle A \rangle_i \wedge \langle B \rangle_j. \tag{2.5.11}$$

Note that the outer product is a grade-raising operation.

Definition 2.28. For arbitrary multivectors A and B we define the *commutator product* as

$$A \times B := \frac{1}{2}(AB - BA). \tag{2.5.12}$$

It is immediately seen that $A \times A = 0$ for any multivector A.

Theorem 2.14. *The set of all k-vectors of even grade, denoted by $\mathcal{C}\ell^{+}_{p,r}$, forms a subalgebra of $\mathcal{C}\ell_{p,r}$ called the* even subalgebra *of $\mathcal{C}\ell_{p,r}$.*

Proof. We need only to show the set is closed under multiplication. This is done easily, since if we multiply any two basis blades of even grade, the number of vectors in the resulting product can only decrease by a multiple of two (every time we reorder the product so that two identical vectors meet, the two together form a scalar). $\square$

Definition 2.29. The *reverse* operator is defined for blades as

$$(\mathbf{x}_1 \mathbf{x}_2 \cdots \mathbf{x}_k)^{\dagger} := \mathbf{x}_k \mathbf{x}_{k-1} \cdots \mathbf{x}_1,$$

and is distributive over addition.

2.5.3 Description of vector reflections

We will now show a simple application of the Clifford algebra to a geometric problem, as presented in [6, p. 85]. We first need some lemmas, the proofs are largely technical.

Lemma 2.15. [16, p. 8-10] *Let* $\mathbf{x}$ *be a vector and* A *a homogeneous multivector from* $\mathcal{C}\ell_{p,r}$*, then*

$$\mathbf{x}A = \mathbf{x} \cdot A + \mathbf{x} \wedge A.$$

□

Lemma 2.16. *Let* $\mathbf{x}, \mathbf{y} \in \mathcal{C}\ell_{p,r}$ *be any two vectors. Then*

$$\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{x} = 0.$$

Proof. This follows from (2.5.4) and the fact that $\mathbf{x} \wedge \mathbf{x} = -\mathbf{x} \wedge \mathbf{x} = 0$. □

Lemma 2.17. *Let* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{C}\ell_{p,r}$ *be any three vectors. Then*

$$\mathbf{z} \cdot (\mathbf{x} \wedge \mathbf{y}) = (\mathbf{x} \cdot \mathbf{z})\mathbf{y} - (\mathbf{y} \cdot \mathbf{z})\mathbf{x} = -(\mathbf{x} \wedge \mathbf{y}) \cdot \mathbf{z}.$$

In words, the inner product of a vector with a bivector yields a vector.

Proof. This is a corollary of a theorem found in [16, p. 9-11]. □

Consider now an arbitrary vector $\mathbf{x}$ and a unit vector $\mathbf{n} \in \mathbb{R}^{p,r}$. Let us denote the component of $\mathbf{x}$ parallel and the component of $\mathbf{x}$ perpendicular to $\mathbf{n}$ as $\mathbf{x}_{\|} = (\mathbf{n} \cdot \mathbf{x})\mathbf{n}$ and $\mathbf{x}_{\perp} = \mathbf{x} - \mathbf{x}_{\|}$, respectively. By expanding the product $\mathbf{nxn}$ using Lemma 2.15, we obtain

$$\mathbf{nxn} = \mathbf{n}(\mathbf{x} \cdot \mathbf{n} + \mathbf{x} \wedge \mathbf{n}) = (\mathbf{x} \cdot \mathbf{n})\mathbf{n} + \mathbf{n} \cdot (\mathbf{x} \wedge \mathbf{n}) + \mathbf{n} \wedge \mathbf{x} \wedge \mathbf{n}.$$

The term $\mathbf{n} \wedge \mathbf{x} \wedge \mathbf{n} = 0$ by Corollary 2.16 and $(\mathbf{x} \cdot \mathbf{n})\mathbf{n} = \mathbf{x}_{\|}$. Applying Lemma 2.17 to $\mathbf{n} \cdot (\mathbf{x} \wedge \mathbf{n})$ yields

$$\mathbf{n} \cdot (\mathbf{x} \wedge \mathbf{n}) = (\mathbf{n} \cdot \mathbf{x})\mathbf{n} - (\mathbf{n} \cdot \mathbf{n})\mathbf{x} = \mathbf{x}_{\|} - \mathbf{x} = -\mathbf{x}_{\perp},$$

therefore

$$\mathbf{nxn} = \mathbf{x}_{\|} - \mathbf{x}_{\perp}.$$

Thus $\mathbf{nxn}$ is a reflection of vector $\mathbf{x}$ by vector $\mathbf{n}$. This result can be used to describe rotations in a space of arbitrary dimension as a composition two successive reflections [6, p. 85-86].

2.5.4 Clifford algebra $\mathcal{C}\ell_{1,1}$

Consider the Minkowski space $\mathbb{R}^{1,1}$ with basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and the Minkowski inner product satisfying $\mathbf{e}_1 \cdot \mathbf{e}_1 = 1 = -\mathbf{e}_2 \cdot \mathbf{e}_2$. The Clifford algebra $\mathcal{C}\ell_{1,1}$ over this space is of dimension 4, with canonical basis

$$\overline{\mathcal{C}\ell_{1,1}} = \{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_{12}\}.$$

The even subalgebra $\mathcal{C}\ell_{1,1}^{+}$ is of dimension 2 with canonical basis

$$\overline{\mathcal{C}\ell_{1,1}^{+}} = \{1, \mathbf{e}_{12}\}$$

and multiplication rules given in the Table 2.1. From the distributivity

	1	$\mathbf{e}_{12}$
1	1	$\mathbf{e}_{12}$
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	1

Table 2.1: Cayley table of basis blades of $\mathcal{C}\ell_{1,1}^{+}$. Rows correspond to the left factor; columns to the right factor.

of geometric product and the relationship between basis terms of $\mathcal{C}\ell_{1,1}^{+}$, we have the following proposition.

Proposition 2.18. *The geometric product of two arbitrary multivectors $A, B \in \mathcal{C}\ell_{1,1}^{+}$ can be written in terms of coordinates as*

$$AB = (a_0 + a_1\mathbf{e}_{12})(b_0 + b_1\mathbf{e}_{12}) = a_0b_0 + a_1b_1 + (a_0b_1 + a_1b_0)\mathbf{e}_{12}. \tag{2.5.13}$$

□

2.5.5 Clifford algebra $\mathcal{C}\ell_{2,1}$

Consider the Minkowski space $\mathbb{R}^{2,1}$ with basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and the Minkowski inner product satisfying $\mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_2 \cdot \mathbf{e}_2 = 1 = -\mathbf{e}_3 \cdot \mathbf{e}_3$. The Clifford algebra $\mathcal{C}\ell_{2,1}$ over this space has dimension 8, with canonical basis

$$\overline{\mathcal{C}\ell_{2,1}} = \{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{23}, \mathbf{e}_{123}\}.$$

	1	$\mathbf{e}_{12}$	$\mathbf{e}_{13}$	$\mathbf{e}_{23}$
1	1	$\mathbf{e}_{12}$	$\mathbf{e}_{13}$	$\mathbf{e}_{23}$
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	-1	$-\mathbf{e}_{23}$	$\mathbf{e}_{13}$
$\mathbf{e}_{13}$	$\mathbf{e}_{13}$	$\mathbf{e}_{23}$	1	$\mathbf{e}_{12}$
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{13}$	$-\mathbf{e}_{12}$	1

Table 2.2: Cayley table of basis blades of $\mathcal{C}\ell_{2,1}^{+}$. Rows correspond to the left factor; columns to the right factor.

The even subalgebra $\mathcal{C}\ell_{2,1}^{+}$ has dimension 4 with canonical basis

$$\overline{\mathcal{C}\ell_{2,1}^{+}} = \{1, \mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{23}\}$$

and multiplication rules given in the Table 2.2.

Note that if we replace the basis vector $\mathbf{e}_3$ with one satisfying $\mathbf{e}_3 \cdot \mathbf{e}_3 = 1$ and thus obtaining the even subalgebra $\mathcal{C}\ell_3^{+}$, this algebra would be isomorphic to the algebra of quaternions. This will allow us to solve equations in $\mathcal{C}\ell_{2,1}^{+}$ by using the methods for solving analogous equations for the quaternions, as long as we note the possibilities of singular cases arising from the indefiniteness of the inner product. This algebra is furthermore isomorphic to the algebra of so-called *split-quaternions*.

Proposition 2.19. *The geometric product of two arbitrary multivectors $A, B \in \mathcal{C}\ell_{2,1}^{+}$ is written in terms of coordinates as*

$$\begin{aligned} AB &= (a_0 + a_1\mathbf{e}_{12} + a_2\mathbf{e}_{13} + a_3\mathbf{e}_{23})(b_0 + b_1\mathbf{e}_{12} + b_2\mathbf{e}_{13} + b_3\mathbf{e}_{23}) \\ &= a_0b_0 - a_1b_1 + a_2b_2 + a_3b_3 \quad + \\ &\quad (a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2)\,\mathbf{e}_{12} + \\ &\quad (a_0b_2 + a_1b_3 + a_2b_0 - a_3b_1)\,\mathbf{e}_{13} + \\ &\quad (a_0b_3 - a_1b_2 + a_2b_1 + a_3b_0)\,\mathbf{e}_{23}. \end{aligned} \tag{2.5.14}$$

□

Proposition 2.20. *The inner product of two homogeneous bivectors $\mathbf{a}, \mathbf{b} \in \mathcal{C}\ell_{2,1}^{+}$ can be written in terms of coordinates as*

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= (a_1\mathbf{e}_{12} + a_2\mathbf{e}_{13} + a_3\mathbf{e}_{23}) \cdot (b_1\mathbf{e}_{12} + b_2\mathbf{e}_{13} + b_3\mathbf{e}_{23}) \\ &= -a_1b_1 + a_2b_2 + a_3b_3 \\ &= \frac{1}{2}(\mathbf{ab} + \mathbf{ba}). \end{aligned} \tag{2.5.15}$$

$\square$

Proposition 2.21. *The commutator product of two homogeneous bivectors* $\mathbf{a}, \mathbf{b} \in \mathcal{C}\ell_{2,1}^{+}$ *is written in terms of coordinates as*

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= (a_1 \mathbf{e}_{12} + a_2 \mathbf{e}_{13} + a_3 \mathbf{e}_{23}) \times (b_1 \mathbf{e}_{12} + b_2 \mathbf{e}_{13} + b_3 \mathbf{e}_{23}) \\ &= (a_2 b_3 - a_3 b_2)\, \mathbf{e}_{12} + (a_1 b_3 - a_3 b_1)\, \mathbf{e}_{13} + (-a_1 b_2 + a_2 b_1)\, \mathbf{e}_{23}. \end{aligned} \tag{2.5.16}$$

$\square$

Propositions 2.20 and 2.21 allow us to rewrite the geometric product of multivectors from $\mathcal{C}\ell_{2,1}^{+}$ in terms of the scalar and bivector parts as

$$AB = (a_0 + \mathbf{a})\,(b_0 + \mathbf{b}) = a_0 b_0 + \mathbf{a} \cdot \mathbf{b} + a_0 \mathbf{b} + b_0 \mathbf{a} + \mathbf{a} \times \mathbf{b}. \tag{2.5.17}$$

Proposition 2.22. *For the product of any two multivectors* $A, B \in \mathcal{C}\ell_{2,1}^{+}$*, the* reversal *acts as:*

$$(AB)^{\dagger} = B^{\dagger} A^{\dagger}.$$

Proof. By direct computation. $\square$

Definition 2.30. For any multivector $A \in \mathcal{C}\ell_{2,1}^{+}$ we define the *trace* as

$$\mathrm{T}(A) = A + A^{\dagger} = 2a_0.$$

Definition 2.31. For any multivector $A \in \mathcal{C}\ell_{2,1}^{+}$ we define the *Clifford norm square* as

$$\mathrm{N}(A) = \mathrm{N}\left(A^{\dagger}\right) = AA^{\dagger} = A^{\dagger}A = a_0^2 + a_1^2 - a_2^2 - a_3^2.$$

We classify multivectors analogous to the causal structure introduced in Definition 2.13 for vectors in Minkowski space as follows:

- If $\mathrm{N}(A) > 0$, A is called a *time-like* multivector,
- If $\mathrm{N}(A) < 0$, A is called a *space-like* multivector,
- If $\mathrm{N}(A) = 0$, A is called a *light-like* multivector.

The next proposition shows that the Clifford norm square is multiplicative.

Proposition 2.23. *For any two multivectors $A, B \in \mathcal{C}\ell_{2,1}^{+}$*

$$\mathrm{N}(AB) = \mathrm{N}(A)\mathrm{N}(B).$$

Proof.

$$\mathrm{N}(AB) = AB(AB)^{\dagger} = ABB^{\dagger}A^{\dagger} = AA^{\dagger}\mathrm{N}(B) = \mathrm{N}(A)\mathrm{N}(B). \qquad \square$$

Definition 2.32. For a multivector $A \in \mathcal{C}\ell_{2,1}^{+}$ which is not light-like we define the *inverse multivector* as

$$A^{-1} = \frac{A^{\dagger}}{\mathrm{N}(A)}.$$

2.5.6 Geometry of $\mathbb{R}^{1,2}$

Consider now $\left\langle \mathcal{C}\ell_{2,1}^{+} \right\rangle_2$, the subspace of all bivectors from $\mathcal{C}\ell_{2,1}^{+}$, together with the Clifford norm defined in 2.31. It is easy to see that this subspace is in fact isomorphic to the Minkowski space $\mathbb{R}^{1,2}$ (also known as the *Minkowski 3-space*), just as the subspace of all pure quaternions is regarded as isomorphic to $\mathbb{R}^3$. As the geometric interpretation of the quadratic equation is a critical part of the solution method used in [13], we shall briefly discuss the structure of $\mathbb{R}^{1,2}$ through the lens of $\left\langle \mathcal{C}\ell_{2,1}^{+} \right\rangle_2$ so we can make use of it in Chapter 4. Note the paradox of the signature of $\mathbb{R}^{1,2}$ being different from the signature the Minkowski space we have initially built the Clifford algebra over.

In this context, we shall use the term "vector" to refer to an element of $\mathbb{R}^{1,2}$, corresponding to some bivector from $\left\langle \mathcal{C}\ell_{2,1}^{+} \right\rangle_2$. Furthermore, all operators applied to such vectors are to be regarded as operators of the Clifford algebra applied to bivectors. Another possible source of confusion might be the behavior of the Clifford inner product. Since we are applying it to bivectors, it turns out that it's value is the negative of the corresponding Minkowski inner product in $\mathbb{R}^{1,2}$. This does not violate the Axiom (C1), as that one holds for vectors from the defining Minkowski space $\mathbb{R}^{2,1}$.

Theorem 2.24. *Every time-like vector $\mathbf{a}$ can be normalized so that $\mathrm{N}(\hat{\mathbf{a}}) = 1$. Every space-like vector $\mathbf{b}$ can be normalized so that $\mathrm{N}(\hat{\mathbf{b}}) = -1$.*

Proof. Take

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{\sqrt{|\mathrm{N}(\mathbf{a})|}}. \qquad \square$$

Definition 2.33. The quantity $\sqrt{|\mathrm{N}(\mathbf{a})|}$ from the previous theorem is called the *magnitude of vector* $\mathbf{a}$.

Proposition 2.25. *For any vector* $\mathbf{a}$*, we have* $\mathbf{a}^2 = \mathbf{a} \cdot \mathbf{a} = -\mathrm{N}(\mathbf{a})$.

Proof. This follows from using Equation (2.5.14), Definition 2.31 and Equation (2.5.15). □

Definition 2.34. A vector $\mathbf{a} = [a_1, a_2, a_3]$ with $\mathrm{N}(\mathbf{a}) = a_1^2 - a_2^2 - a_3^2 = \pm 1$ is called a *positive* or *negative unit vector*, respectively. The sets of all positive and all negative unit vectors form the *positive unit hyperboloid* with two sheets and the *negative unit hyperboloid* with one sheet. The set of all light-like vectors forms the *null* or *light cone.* The region of the light cone with $a_1 > 0$ is called the *future cone*, the region of the light cone with $a_1 < 0$ is called the *past cone.* See Figure 2.2 for illustration.

Definition 2.35. Two vectors $\mathbf{a} = [a_1, a_2, a_3]$ and $\mathbf{b} = [b_1, b_2, b_3]$ are said to be *orthogonal* if $\mathbf{a} \cdot \mathbf{b} = 0$.

It is immediately seen that for any given vector $\mathbf{a} = [a_1, a_2, a_3]$, the subspace of all orthogonal vectors forms a plane with normal vector $[-a_1, a_2, a_3]$ in the Euclidean sense. This means that for a light-like vector, the orthogonal plane is contains the vector itself and is tangent to the light cone.

Theorem 2.26. *A set of nonzero orthogonal vectors of which neither is light-like is linearly independent.*

Proof. The standard proof by contradiction from linear algebra suffices, note however the necessity of all vectors being time-like or space-like. □

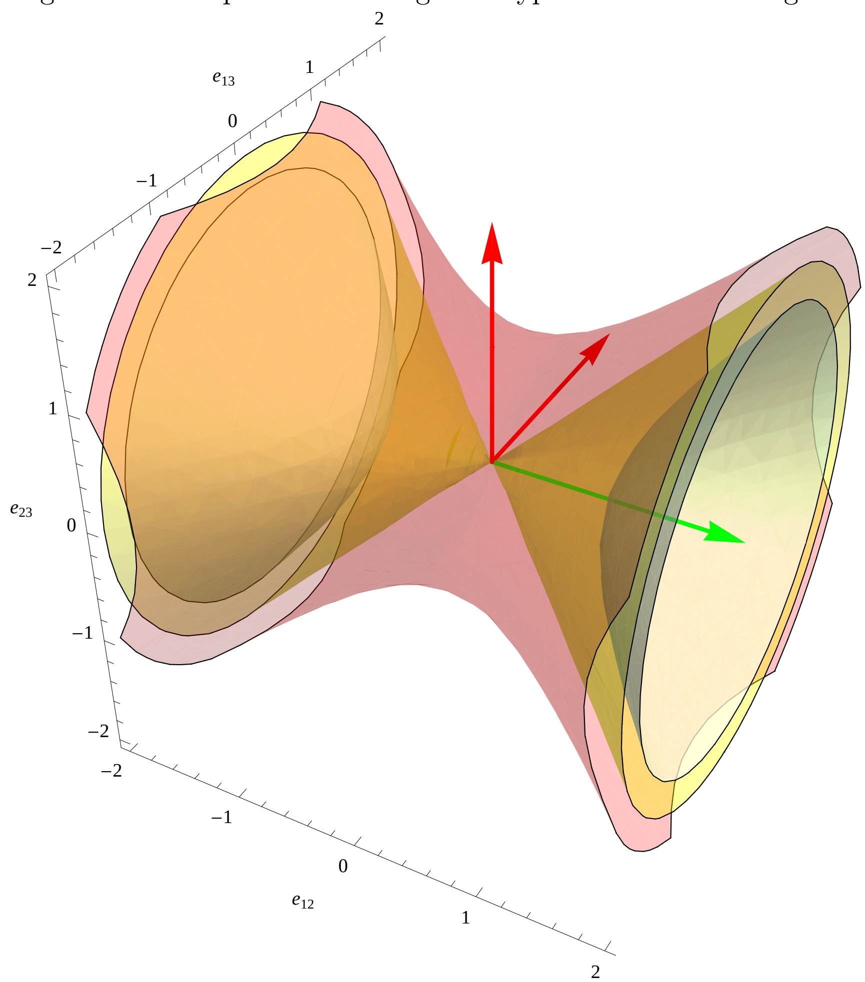
Theorem 2.27. *The commutator product of two vectors* $\mathbf{a} \times \mathbf{b}$ *is orthogonal to both* $\mathbf{a}$ *and* $\mathbf{b}$.

Proof. Using Definition 2.28 and result established for bivectors in Proposition 2.20, we get for, say, the vector $\mathbf{a}$:

$$\begin{aligned}\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} &= \frac{1}{2}\left(\frac{1}{2}(\mathbf{ab} - \mathbf{ba})\mathbf{a} + \frac{1}{2}\mathbf{a}(\mathbf{ab} - \mathbf{ba})\right) \\ &= \frac{1}{4}\left(\mathbf{aba} - \mathbf{ba}^2 + \mathbf{a}^2\mathbf{b} - \mathbf{aba}\right) = 0.\end{aligned}$$

□

Figure 2.2: The positive and negative hyperboloid and the light cone.



The proof for **b** is similar.

Corollary 2.1. If any two vectors are coplanar with a light-like vector, their commutator is a multiple of that light-like vector.

Theorem 2.28 (Pythagorean Theorem)**.** *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be two orthogonal vectors. Then* $\mathrm{N}(\mathbf{a}+\mathbf{b}) = \mathrm{N}(\mathbf{a}) + \mathrm{N}(\mathbf{b})$.

Proof.

$$\begin{aligned}\mathrm{N}(\mathbf{a}+\mathbf{b}) &= (\mathbf{a}+\mathbf{b})(-\mathbf{a}-\mathbf{b}) = \mathrm{N}(\mathbf{a}) - \mathbf{ab} - \mathbf{ba} + \mathrm{N}(\mathbf{b}) \\ &= \mathrm{N}(\mathbf{a}) - 2(\mathbf{a}\cdot\mathbf{b}) + \mathrm{N}(\mathbf{b}) = \mathrm{N}(\mathbf{a}) + \mathrm{N}(\mathbf{b}).\end{aligned}$$ □

Theorem 2.29. *For any time-like vector $\mathbf{a} = [a_1, a_2, a_3]$, all vectors orthogonal to $\mathbf{a}$ are space-like. Conversely, for any two linearly independent space-like vectors $\mathbf{b}$ and $\mathbf{c}$ generating a space-like plane, their commutator product $\mathbf{b}\times\mathbf{c}$ is time-like.*

Proof. If $\mathbf{a} = \mathbf{e}_{12}$, then obviously the orthogonal subspace is spanned by $\mathbf{e}_{13}$ and $\mathbf{e}_{23}$, which generate only space-like vectors. Assume now that $\mathbf{a}$ and $\mathbf{e}_{12}$ are linearly independent and let us denote $\mathrm{N}(\mathbf{a}) = n > 0$. Then $\mathbf{a}\times\mathbf{e}_{12} = [0, -a_3, a_2]$ is orthogonal to $\mathbf{a}$ and

$$\mathrm{N}\left(\mathbf{a}\times\mathbf{e}_{12}\right) = -a_2^2 - a_3^2 < 0.$$

Furthermore, $(\mathbf{a}\times\mathbf{e}_{12})\times\mathbf{a} = \left[-a_2^2 - a_3^2, -a_1a_2, -a_1a_3\right]$ is orthogonal to both $\mathbf{a}$ and $\mathbf{a}\times\mathbf{e}_{12}$. Observe that $-a_2^2 - a_3^2 = n - a_1^2$, and

$$\begin{aligned}\mathrm{N}\Big((\mathbf{a}\times\mathbf{e}_{12})\times\mathbf{a}\Big) &= n^2 - 2na_1^2 + a_1^4 - a_2^2 - a_3^2 \\ &= n^2 - 2na_1^2 + a_1^2\big(\underbrace{a_1^2 - a_2^2 - a_3^2}_{=n}\big) \\ &= n\left(n - a_1^2\right) \\ &= n\left(-a_2^2 - a_3^2\right) < 0.\end{aligned}$$

The proof is completed by invoking the Pythagorean Theorem on any linear combination of $\mathbf{a}\times\mathbf{e}_{12}$ and $(\mathbf{a}\times\mathbf{e}_{12})\times\mathbf{a}$.

To prove the converse statement, assume for contradiction that $\mathbf{b}\times\mathbf{c}$ is not time-like. $\mathbf{b}\times\mathbf{c}$ cannot be light-like, because then the plane orthogonal to $\mathbf{b}\times\mathbf{c}$ would contain $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{b}\times\mathbf{c}$, however $\mathbf{b}$ and $\mathbf{c}$ generate a space-like plane. Hence, $\mathbf{b}\times\mathbf{c}$ must be space-like. It follows that $(\mathbf{b}\times\mathbf{c})\times\mathbf{b}$ must be time-like, since otherwise we would have a set of 3 orthogonal linearly independent vectors and thus a basis (Theorem 2.26), which would however not generate time-like vectors as a consequence of Pythagorean

Theorem. However, if we now denote $\mathrm{N}(\mathbf{b}) := n = b_1^2 - b_2^2 - b_3^2$, we have

$$(\mathbf{b} \times \mathbf{c}) \times \mathbf{b} = \begin{bmatrix} -b_2^2 c_1 - b_3^2 c_1 + b_2 c_2 b_1 + b_3 c_3 b_1 \\ -b_1 c_1 b_2 + b_1^2 c_2 - b_3^2 c_2 + b_3 c_3 b_2 \\ -b_1 c_1 b_3 + b_2 c_2 b_3 + b_1^2 c_3 - b_2^2 c_3 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \left(n - b_1^2\right) c_1 + \left(b_2 c_2 + b_3 c_3\right) b_1 \\ -b_1 c_1 b_2 + \left(n + b_2^2\right) c_2 + b_3 c_3 b_2 \\ \left(-b_1 c_1 + b_2 c_2\right) b_3 + \left(n + b_3^2\right) c_3 \end{bmatrix}^{\mathrm{T}}$$

$$= \begin{bmatrix} \left(-b_1 c_1 + b_2 c_2 + b_3 c_3\right) b_1 + n c_1 \\ \left(-b_1 c_1 + b_2 c_2 + b_3 c_3\right) b_2 + n c_2 \\ \left(-b_1 c_1 + b_2 c_2 + b_3 c_3\right) b_3 + n c_3 \end{bmatrix}^{\mathrm{T}} = (\mathbf{b} \cdot \mathbf{c})\mathbf{b} + n\mathbf{c},$$

in words, $\mathbf{b}$ and $\mathbf{c}$ generate a time-like vector $(\mathbf{b} \times \mathbf{c}) \times \mathbf{b}$, which is a contradiction. □

Theorem 2.30. *Let $\mathbf{a} = [a_1, a_2, a_3]$ be any vector, $\mathbf{x}_+$ any positive unit vector and $\mathbf{x}_-$ any negative unit vector. Then*

$$\mathbf{a} = \begin{cases} -\left(\mathbf{x}_+ \cdot \mathbf{a}\right) \mathbf{x}_+ + \left(\mathbf{x}_+ \times \mathbf{a}\right) \times \mathbf{x}_+, \\ \left(\mathbf{x}_- \cdot \mathbf{a}\right) \mathbf{x}_- - \left(\mathbf{x}_- \times \mathbf{a}\right) \times \mathbf{x}_-. \end{cases} \tag{2.5.18}$$

Proof. For any vector $\mathbf{x} = [x_1, x_2, x_3]$ we have

$$(\mathbf{x} \cdot \mathbf{a})\,\mathbf{x} = \begin{bmatrix} -x_1^2 a_1 + x_1 x_2 a_2 + x_1 x_3 a_3 \\ -x_1 x_2 a_1 + x_2^2 a_2 + x_2 x_3 a_3 \\ -x_1 x_3 a_1 + x_2 x_3 a_2 + x_3^2 a_3 \end{bmatrix}^{\mathrm{T}},$$

and

$$(\mathbf{x} \times \mathbf{a}) \times \mathbf{x} = \begin{bmatrix} x_1 x_3 a_3 - x_3^2 a_1 + x_1 x_2 a_2 - x_2^2 a_1 \\ x_2 x_3 a_3 - x_3^2 a_2 + x_1^2 a_2 - x_1 x_2 a_1 \\ -x_2^2 a_3 + x_2 x_3 a_2 + x_1^2 a_3 - x_1 x_3 a_1 \end{bmatrix}^{\mathrm{T}}.$$

Subtracting the two terms therefore yields

$$(\mathbf{x} \cdot \mathbf{a})\,\mathbf{x} - (\mathbf{x} \times \mathbf{a}) \times \mathbf{x} = \left(-x_1^2 + x_2^2 + x_3^2\right) \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}^{\mathrm{T}} = \begin{cases} -\mathbf{a} & \text{if } \mathrm{N}(\mathbf{x}) = 1, \\ 0 & \text{if } \mathrm{N}(\mathbf{x}) = 0, \\ \mathbf{a} & \text{if } \mathrm{N}(\mathbf{x}) = -1. \end{cases}$$

□

Definition 2.36. Let $\mathbf{a}$ and $\mathbf{b}$ be space-like vectors. If the plane spanned by $\mathbf{a}$ and $\mathbf{b}$ contains only space-like vectors, we define the *circular angle* between $\mathbf{a}$ and $\mathbf{b}$ as a number $\phi \in [0, \pi]$ satisfying

$$\cos\phi = \frac{\mathbf{a}\cdot\mathbf{b}}{\sqrt{\mathrm{N}(\mathbf{a})\mathrm{N}(\mathbf{b})}}, \qquad \sin\phi = \sqrt{\frac{\mathrm{N}(\mathbf{a}\times\mathbf{b})}{\mathrm{N}(\mathbf{a})\mathrm{N}(\mathbf{b})}}.$$

The next theorem shows that the Pythagorean Theorem holds for circular angles.

Theorem 2.31. *Let ϕ be a circular angle between $\mathbf{a}$ and $\mathbf{b}$ as defined in 2.36, then* $\sin^2\phi + \cos^2\phi = 1$.

Proof. Recall Propositions 2.22 and 2.25. We get

$$\begin{aligned}
\sin^2\phi + \cos^2\phi &= \frac{\left(\frac{1}{2}(\mathbf{ab}+\mathbf{ba})\right)^2 + \frac{1}{2}(\mathbf{ab}-\mathbf{ba})\frac{1}{2}(\mathbf{ab}-\mathbf{ba})^\dagger}{\mathrm{N}(\mathbf{a})\mathrm{N}(\mathbf{b})}\\
&= \frac{\frac{1}{4}\left(\mathbf{abab}+2\mathbf{a}^2\mathbf{b}^2+\mathbf{baba}\right) + \frac{1}{4}(\mathbf{ab}-\mathbf{ba})(\mathbf{ba}-\mathbf{ab})}{\mathrm{N}(\mathbf{a})\mathrm{N}(\mathbf{b})}\\
&= \frac{\frac{1}{4}\left(\mathbf{abab}+2\mathbf{a}^2\mathbf{b}^2+\mathbf{baba}\right) + \frac{1}{4}\left(2\mathbf{a}^2\mathbf{b}^2-\mathbf{abab}-\mathbf{baba}\right)}{\mathrm{N}(\mathbf{a})\mathrm{N}(\mathbf{b})}\\
&= \frac{\mathbf{a}^2\mathbf{b}^2}{\mathrm{N}(\mathbf{a})\mathrm{N}(\mathbf{b})} = 1.
\end{aligned}$$

□

Definition 2.37. Let $\mathbf{a}$ and $\mathbf{b}$ be both time-like or both space-like vectors. We define the *hyperbolic angle* between $\mathbf{a}$ and $\mathbf{b}$ as a number $\psi \in (-\infty, +\infty)$ satisfying

$$\cosh_{\mathbb{R}}\psi = \frac{\left|\mathbf{a}\cdot\mathbf{b}^\dagger\right|}{\sqrt{\mathrm{N}(\mathbf{a})\mathrm{N}(\mathbf{b})}},$$
$$\sinh\psi = \sqrt{\frac{|\mathrm{N}(\mathbf{a}\times\mathbf{b})|}{\mathrm{N}(\mathbf{a})\mathrm{N}(\mathbf{b})}}.$$

By $\cosh_{\mathbb{R}}$ we denote the standard function cosh with range $[1, +\infty)$. To simplify our notation however, we will allow cosh as a function of two vectors to have negative values; this will correspond to the case of $\mathbf{a}$ and $\mathbf{b}$ lying on opposite branches of the same hyperbola.

Theorem 2.32. *Let ψ be a hyperbolic angle between $\mathbf{a}$ and $\mathbf{b}$ as defined in 2.37, then* $\cosh^2\psi - \sinh^2\psi = 1$.

Proof. Using the same computations as in Theorem 2.31, we get

$$\cosh^2\psi - \sinh^2\psi = \frac{\frac{1}{4}\left(\mathbf{abab} + 2\mathbf{a}^2\mathbf{b}^2 + \mathbf{baba}\right) - \frac{1}{4}\left|2\mathbf{a}^2\mathbf{b}^2 - \mathbf{abab} - \mathbf{baba}\right|}{\mathrm{N}(\mathbf{a})\mathrm{N}(\mathbf{b})}.$$

Because Proposition 2.29 gives $\mathrm{N}(\mathbf{a}\times\mathbf{b}) < 0$, we can, remove the absolute value and obtain

$$\cosh^2\psi - \sinh^2\psi = \frac{\frac{1}{4}\left(\mathbf{abab} + 2\mathbf{a}^2\mathbf{b}^2 + \mathbf{baba}\right) + \frac{1}{4}\left(2\mathbf{a}^2\mathbf{b}^2 - \mathbf{abab} - \mathbf{baba}\right)}{\mathrm{N}(\mathbf{a})\mathrm{N}(\mathbf{b})} = 1. \quad \square$$

These definitions allow us to state an equivalent to Equation (2.4.3). Let $X = [x_0, \mathbf{x}]$ be a positive unit multivector and $\hat{\mathbf{x}}$ be the bivector $\mathbf{x}$ normalized in the sense of Theorem 2.24, if possible. Then we can write X as

$$X = \begin{cases} \left[\cos\frac{\phi}{2}, \sin\frac{\phi}{2}\hat{\mathbf{x}}\right], \quad \phi \in (-\pi, \pi] & \text{if } \mathbf{x} \text{ is time-like,} \\ \left[\cosh\frac{\psi}{2}, \sinh\frac{\psi}{2}\hat{\mathbf{x}}\right], \quad \psi \in \mathbb{R} & \text{if } \mathbf{x} \text{ is space-like,} \\ [\pm 1, \mathbf{x}] & \text{if } \mathbf{x} \text{ is light-like.} \end{cases} \tag{2.5.19}$$

If we let X be a negative unit multivector and $\hat{\mathbf{x}}$ be the bivector $\mathbf{x}$ normalized in the sense of Theorem 2.24, we can write X as

$$X = \left[\sinh\frac{\psi}{2}, \cosh\frac{\psi}{2}\hat{\mathbf{x}}\right], \quad \psi \in \mathbb{R}. \tag{2.5.20}$$

Finally, if X is a lightlike multivector, we can write X as

$$X = \begin{cases} \left[\pm\sqrt{|\mathrm{N}(\mathbf{x})|}, \mathbf{x}\right] & \text{if } \mathbf{x} \text{ is space-like,} \\ [0, \mathbf{x}] & \text{if } \mathbf{x} \text{ is light-like.} \end{cases} \tag{2.5.21}$$

Chapter 3

Quadratic equations with left coefficients

As the algebraic product in Clifford algebra is non-commutative, a quadratic equation with the variable and the coefficients being multivectors in the most general form without employing the reversal (†) operator is

$$AXBXC + DXE + F = 0.$$

In this chapter, we will present a discussion about solving a special case of this equation, namely a monic, unilateral equation with coefficients to the left of X. This will be done for two particular even Clifford subalgebras over the Minkowski spaces $\mathbb{R}^{1,1}$ and $\mathbb{R}^{2,1}$, which arise from the study of Medial Axis Transform as we mentioned in the introductory chapter.

3.1 Left-sided equation in $\mathcal{C}\ell_{1,1}^+$

Consider the equation

$$X^2 + BX + C = 0, \quad X, B, C \in \mathcal{C}\ell_{1,1}^+, \tag{3.1.1}$$

where X is an unknown multivector. Writing the first two terms on the left in expanded form we have

$$
\begin{aligned}
X^2 &= (x_0 + x_1\mathbf{e}_{12})(x_0 + x_1\mathbf{e}_{12}) = x_0^2 + 2x_0x_1\mathbf{e}_1\mathbf{e}_2 + x_1^2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 \\
&= x_0^2 + 2x_0x_1\mathbf{e}_1\mathbf{e}_2 - x_1^2\underbrace{\mathbf{e}_1\mathbf{e}_1}_{=1}\underbrace{\mathbf{e}_2\mathbf{e}_2}_{=-1} = x_0^2 + x_1^2 + 2x_0x_1\mathbf{e}_{12},
\end{aligned}
\tag{3.1.2}
$$

and

$$
BX = (b_0 + b_1\mathbf{e}_{12})(x_0 + x_1\mathbf{e}_{12}) = b_0x_0 + b_1x_1 + (b_0x_1 + b_1x_0)\,\mathbf{e}_{12}. \tag{3.1.3}
$$

Collecting the like terms, we obtain a system of equations

$$
\begin{aligned}
x_0^2 + x_1^2 + b_0x_0 + b_1x_1 + c_0 &= 0, \\
2x_0x_1 + b_0x_1 + b_1x_0 + c_1 &= 0,
\end{aligned}
$$

which can be rewritten in the form

$$
\left(x_0 + \frac{b_0}{2}\right)^2 + \left(x_1 + \frac{b_1}{2}\right)^2 = \frac{b_0^2 + b_1^2}{4} - c_0, \tag{3.1.4a}
$$

$$
x_0 + \frac{b_0}{2} = \frac{\frac{b_0b_1 - 2c_1}{4}}{x_1 + \frac{b_1}{2}}. \tag{3.1.4b}
$$

Geometrically, this system of equations describes intersection of a circle (3.1.4a) and a hyperbola (3.1.4b) in the Euclidean plane, both centered at the point with coordinates $\left(-\frac{b_0}{2}, -\frac{b_1}{2}\right)$. Therefore, Equation (3.1.1) has either 4, 2 or no solutions.

3.2 Left-sided equation in $\mathcal{C}\ell_{2,1}^{+}$

Consider the equation

$$
X^2 + BX + C = 0, \quad X, B, C \in \mathcal{C}\ell_{2,1}^{+}, \tag{3.2.1}
$$

where X is an unknown multivector. In coordinates, this equation is a system of four quadratic equations for the four coordinates of X. The earliest complete study of solutions to the equation analogous to (3.2.1) in the quaternions is found in [20], this is also the method we chose in this thesis. Numerous other studies, both analytical and numerical, emerged especially in the 2000's, many being cited in, for example, [8]. Of these, we will point out [18], which presents a quaternion analogue of the quadratic formula.

3.2.1 Degenerate case

Firstly let us discuss the degenerate case $B = \langle B \rangle = b_0$ and $C = \langle C \rangle = c_0$. Using Equation (2.5.17), we can split Equation (3.2.1) into a system of equations for the scalar part x_0 and bivector part $\mathbf{x}$ as

$$x_0^2 + \mathbf{x} \cdot \mathbf{x} + b_0 x_0 + c_0 = 0, \tag{3.2.2a}$$

$$(2x_0 + b_0)\,\mathbf{x} = \vec{0}. \tag{3.2.2b}$$

Equation (3.2.2b) immediately yields two cases:

- $\mathbf{x} = \vec{0}$. In this case the equation (3.2.2a) becomes

$$x_0^2 + b_0 x_0 + c_0 = 0, \tag{3.2.3}$$

  which yields the solutions $X_{1,2} = \frac{b_0 \pm \sqrt{b_0^2 - 4c_0}}{2}$ if $b_0^2 - 4c_0 \geq 0$ and no solutions otherwise.

- $2x_0 + b_0 = 0$. In this case we can rewrite (3.2.2a) as

$$\frac{b_0^2}{4} + x_1^2 = x_2^2 + x_3^2 + c_0. \tag{3.2.4}$$

  This equation admits a two-parameter family of eight solutions of the form

$$X = -\frac{b_0}{2} \pm u\,\mathbf{e}_{12} \pm v\,\mathbf{e}_{13} \pm \sqrt{\frac{b_0^2}{4} + u^2 - v^2 - c_0}\,\mathbf{e}_{23}, \quad u, v \in M, \tag{3.2.5}$$

  where

$$M = \left\{ u, v \in \mathbb{R}^2 : \frac{b_0^2}{4} + u^2 - v^2 - c_0 \geq 0 \right\}, \tag{3.2.6}$$

  which, in other words, is the interior of a hyperbola given by the equation

$$\frac{u^2}{c_0 - \frac{b_0^2}{4}} - \frac{v^2}{c_0 - \frac{b_0^2}{4}} = 1. \tag{3.2.7}$$

3.2.2 Non-degenerate case

We will now investigate the non-degenerate case using and expanding on the approach of Niven [20].

Proposition 3.1. *For any multivector $X \in \mathcal{C}\ell_{2,1}^{+}$ the equality*

$$X^2 - \mathrm{T}(X)X + \mathrm{N}(X) = 0$$

always holds.

Proof. This follows by direct computation. □

We may, without loss of generality, assume that in (3.2.1), $\mathrm{T}(B) = 0$; otherwise we use the substitution $X = Y - \frac{1}{2}\mathrm{T}(B)$, which yields the equation with this property. If we now factor out $X^2 - \mathrm{T}(X)X + \mathrm{N}(X)$ on the left hand side of Equation (3.2.1), we obtain

$$X^2 + BX + C = X^2 - \mathrm{T}(X)X + \mathrm{N}(X) + \underbrace{(B + \mathrm{T}(X))}_{:=F} X + \underbrace{C - \mathrm{N}(X)}_{:=G}. \tag{3.2.8}$$

Since $X^2 - \mathrm{T}(X)X + \mathrm{N}(X)$ vanishes by Proposition 3.1, we only need solve the equation

$$FX + G = 0. \tag{3.2.9}$$

This implies two cases to investigate

- F is lightlike and thus has no inverse.
- F has an inverse; thus we search for $X = -F^{-1}G$.

F is lightlike

Assuming F is lightlike and recalling we assumed $\mathrm{T}(B) = 0$, we obtain that

$$FF^{\dagger} = (B + \mathrm{T}(X))(B^{\dagger} + \mathrm{T}(X)) = \mathrm{N}(B) + \mathrm{T}^2(X) = 0. \tag{3.2.10}$$

Now, if the coefficient B is time-like, this case yields no solution. Otherwise, assuming B is not time-like, we now know that $x_0 = \pm\frac{\sqrt{|N(B)|}}{2} := t$. If we now rewrite the equation (3.2.8) into components, we obtain:

$$\begin{aligned}
-x_1^2 - b_1x_1 + x_2^2 + b_2x_2 + x_3^2 + b_3x_3 &= -c_0 - 2t^2, \\
2tx_1 - b_3x_2 + b_2x_3 &= -c_1 - tb_1, \\
-b_3x_1 + 2tx_2 + b_1x_3 &= -c_2 - tb_2, \\
b_2x_1 - b_1x_2 + 2tx_3 &= -c_3 - tb_3.
\end{aligned} \tag{3.2.11}$$

As this system of equations is overdetermined, it suffices to only solve the system of three linear equations and verify whether or not this solution also solves the first equation.

F is not lightlike

Assuming F is not lightlike, we can solve the equation (3.2.8) by setting

$$X = -F^{-1}G = -\frac{F^\dagger G}{\mathrm{N}(F)}. \tag{3.2.12}$$

Note that the right hand side is in terms of known coefficients B and C and unknowns $\mathrm{T}(X)$ and $\mathrm{N}(X)$; once we find these unknowns, we are able to contruct the solution X using this relationship. Furthermore

$$X^\dagger = -\frac{G^\dagger F}{\mathrm{N}(F)}, \tag{3.2.13}$$

from which we obtain

$$t := \mathrm{T}(X) = -\frac{F^\dagger G + G^\dagger F}{\mathrm{N}(F)}, \tag{3.2.14a}$$

$$n := \mathrm{N}(X) = \frac{F^\dagger G G^\dagger F}{\mathrm{N}^2(F)} = \frac{\mathrm{N}(G)}{\mathrm{N}(F)}, \tag{3.2.14b}$$

or equivalently

$$\begin{aligned} t\mathrm{N}(F) + F^\dagger G + G^\dagger F &= 0, \\ n\mathrm{N}(F) - \mathrm{N}(G) &= 0, \end{aligned}$$

which expands into

$$\begin{aligned} t^3 - 2tn + (\mathrm{N}(B) + \mathrm{T}(C))t + B^\dagger C + C^\dagger B &= 0, \\ -n^2 + nt^2 + (\mathrm{N}(B) + \mathrm{T}(C))n - \mathrm{N}(C) &= 0. \end{aligned}$$

Note that $B^\dagger C + C^\dagger B$ is the trace of $B^\dagger C$ (or $C^\dagger B$) and is thus real. Let us define

$$\kappa :\,= \mathrm{N}(B) + \mathrm{T}(C), \tag{3.2.15a}$$

$$\lambda :\,= B^\dagger C + C^\dagger B, \tag{3.2.15b}$$

$$\mu :\,= \mathrm{N}(C). \tag{3.2.15c}$$

This allows us to arrive at the system of two real polynomial equations with unknowns t and n:

$$t^3 - 2tn + \kappa t + \lambda = 0, \tag{3.2.16a}$$

$$-n^2 + nt^2 + \kappa n - \mu = 0. \tag{3.2.16b}$$

If $\lambda = 0$, this system admits extra solution(s) with $t = 0$ (the remaining roots of (3.2.16a), if they exist, are covered by the case $\lambda \neq 0$) and thus (3.2.16b) becomes

$$-n^2 + \kappa n - \mu = 0, \tag{3.2.17}$$

which yields none, one or two solutions.

Otherwise, assuming $t \neq 0$, from (3.2.16a) we can express n as

$$n = \frac{t^3 + \kappa t + \lambda}{2t}. \tag{3.2.18}$$

Substituting into (3.2.16b) we obtain

$$t^6 + 2\kappa t^4 + (\kappa^2 - 4\mu)t^2 - \lambda^2 = 0. \tag{3.2.19}$$

This is a cubic in t^2; we are looking for positive real roots of which there may be from none up to three; each such root will yield two solutions for t and n.

3.2.3 Summary

For a degenerate case with coefficients B, C being pure scalars, we have a two-parameter family of eight solutions of the form (3.2.5). Furthermore, we may obtain up to two scalar solutions from the Equation (3.2.3).

For a non-degenerate case, we use the form (3.2.8); firstly we check if a solution is possible for a light-like F by solving the system of Equations (3.2.11), which may yield zero, one, two or a one-parameter family of solutions (however since the system is overdetermined, this would be very unlikely).

Then, assuming F is not lightlike, using substitutions (3.2.15), we obtain the system (3.2.16) for the trace t and norm n of a possible solution. From (3.2.19) along with (3.2.18), we may obtain up to six such pairs; furthermore, if the quantity (3.2.15b) vanishes, we may obtain up to two extra homogeneous bivector solutions from (3.2.17). All the combinations of t and n obtained by these methods can be substituted into (3.2.12) to receive the actual solution X.

Chapter 4

Solutions to $X\mathbf{a}X^{\dagger} = \mathbf{d}$.

Consider the equation

$$X\mathbf{a}X^{\dagger} = \mathbf{d}, \tag{4.0.1}$$

where $\mathbf{a}$ and $\mathbf{d}$ are pure bivectors from $\mathcal{C}\ell_{2,1}^{+}$. The equation of this type has been analyzed from the point of view of linear transformations in the algebra of split quaternions ([21]). However, these analyses often focus only on the case where the element corresponding to X is time-like and don't provide discussion on the number and structure of solutions. Our goal is to present a complete, self contained study that includes and explicitly describes solutions to all cases, employing the tools Clifford algebra provides us with. To this end, we will adapt the solution procedure for this equation in quaternions, which can be found, for instance, in [13], [12] and [9].

Examining this equation in quaternions, it shows that if we identify $\mathbf{a}$ and $\mathbf{d}$ with points on a unit sphere in $\mathbb{R}^3$, the solution is a family of quaternion rotations (see Equation (2.4.5)) with vector parts on the intersection of the unit sphere and a plane bisecting the angle between $\mathbf{a}$ and $\mathbf{d}$. We shall now invoke the isomorphism to Minkowski space $\mathbb{R}^{1,2}$ we explored in Subsection 2.5.6, which, in contrast to the quaternion case, brings a multitude of separate and singular cases. However, the solutions will still ultimately be found on intersections of a bisecting plane with a particular unit surface.

Theorem 4.1. *Equation* (4.0.1) *has no solutions if* $\operatorname{sgn} \mathrm{N}(\mathbf{a}) = -\operatorname{sgn} \mathrm{N}(\mathbf{d}) \neq 0$.

Proof. This can be immediately seen if we take the Clifford norm of both sides and recall Proposition 2.23:

$$\mathrm{N}\left(X\mathbf{a}X^\dagger\right) = \mathrm{N}^2(X)\mathrm{N}(\mathbf{a}) = \mathrm{N}(\mathbf{d}). \qquad \square$$

The proof of the previous theorem also gives us an important general property of $\mathrm{N}(X)$ if $\mathrm{N}(\mathbf{a})\mathrm{N}(\mathbf{d}) \neq 0$. Namely,

$$\mathrm{N}(X) = \pm\sqrt{\frac{\mathrm{N}(\mathbf{d})}{\mathrm{N}(\mathbf{a})}}. \tag{4.0.2}$$

This allows us to transform Equation (4.0.1) using the substitutions

$$\widehat{X} = \frac{X}{\sqrt[4]{\frac{\mathrm{N}(\mathbf{d})}{\mathrm{N}(\mathbf{a})}}}, \qquad \hat{\mathbf{a}} = \frac{\mathbf{a}}{\sqrt{|\mathrm{N}(\mathbf{a})|}}, \qquad \hat{\mathbf{d}} = \frac{\mathbf{d}}{\sqrt{|\mathrm{N}(\mathbf{d})|}},$$

into

$$\widehat{X}\hat{\mathbf{a}}\widehat{X}^\dagger = \hat{\mathbf{d}}, \tag{4.0.3}$$

where $\widehat{X}$, $\hat{\mathbf{a}}$ and $\hat{\mathbf{d}}$ are all unit multivectors.

Before examining case-specific solutions to the above equation, let us prove a few useful lemmas.

Lemma 4.2. *For any two vectors* $\mathbf{a}$ *and* $\mathbf{b}$

$$\mathrm{N}(\mathbf{a} \pm \mathbf{b}) = \mathrm{N}(\mathbf{a}) + \mathrm{N}(\mathbf{b}) \mp 2(\mathbf{a} \cdot \mathbf{b}).$$

Proof.

$$\mathrm{N}(\mathbf{a} \pm \mathbf{b}) = (\mathbf{a} \pm \mathbf{b})(-\mathbf{a} \mp \mathbf{b}) = \mathrm{N}(\mathbf{a}) + \mathrm{N}(\mathbf{b}) \mp \mathbf{ab} \mp \mathbf{ba} = \mathrm{N}(\mathbf{a}) + \mathrm{N}(\mathbf{b}) \mp 2(\mathbf{a} \cdot \mathbf{b}). \quad \square$$

Lemma 4.3. *For any two vectors* $\mathbf{a}$ *and* $\mathbf{b}$

$$\mathrm{N}(\mathbf{a} \times \mathbf{b}) = \mathrm{N}(\mathbf{a})\mathrm{N}(\mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})^2.$$

Proof. Expanding the expression according to Definitions 2.31 and 2.28, we get

$$\mathrm{N}(\mathbf{a} \times \mathbf{b}) = \frac{1}{4}\left(2\mathrm{N}(\mathbf{a})\mathrm{N}(\mathbf{b}) - (\mathbf{ab})^2 - (\mathbf{ba})^2\right).$$

Furthermore using Proposition 2.25 we have

$$(\mathbf{ab})^2 = (\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \times \mathbf{b})^2 = (\mathbf{a} \cdot \mathbf{b})^2 - \mathrm{N}(\mathbf{a} \times \mathbf{b}) + 2(\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \times \mathbf{b}),$$
$$(\mathbf{ba})^2 = (\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \times \mathbf{a})^2 = (\mathbf{a} \cdot \mathbf{b})^2 - \mathrm{N}(\mathbf{a} \times \mathbf{b}) - 2(\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \times \mathbf{b}),$$

and therefore

$$\begin{aligned}
\mathrm{N}(\mathbf{a}\times\mathbf{b}) &= \frac{1}{2}\mathrm{N}(\mathbf{a})\mathrm{N}(\mathbf{b}) - \frac{1}{2}(\mathbf{a}\cdot\mathbf{b})^2 + \frac{1}{2}\mathrm{N}(\mathbf{a}\times\mathbf{b}),\\
\mathrm{N}(\mathbf{a}\times\mathbf{b}) &= \mathrm{N}(\mathbf{a})\mathrm{N}(\mathbf{b}) - (\mathbf{a}\cdot\mathbf{b})^2.
\end{aligned}$$ □

Corollary 4.1. If a vector $\mathbf{c}$ is orthogonal to a $\mathbf{a}-\mathbf{b}$ and $\mathrm{N}(\mathbf{a}) = \mathrm{N}(\mathbf{b})$, then $\mathrm{N}(\mathbf{c}\times\mathbf{a}) = \mathrm{N}(\mathbf{c}\times\mathbf{b})$.

Proof. We immediately see that $\mathbf{c}\cdot(\mathbf{a}-\mathbf{b}) = 0$ implies $\mathbf{c}\cdot\mathbf{a} = \mathbf{c}\cdot\mathbf{b}$. □

Lemma 4.4. *For a unit vector* $\mathbf{a}$ *and a unit or light-like vector* $\mathbf{b}$

$$\mathrm{N}(\mathbf{a}\times\mathbf{b}) = \mathrm{N}\big((\mathbf{a}\times\mathbf{b})\times\mathbf{a}\big)$$

if $\mathbf{a}$ *is time-like and*

$$\mathrm{N}(\mathbf{a}\times\mathbf{b}) = -\mathrm{N}\big((\mathbf{a}\times\mathbf{b})\times\mathbf{a}\big)$$

if $\mathbf{a}$ *is space-like.*

Proof. Expanding the expression on the right hand side according to Definitions 2.31 and 2.28, we get

$$\begin{aligned}
\mathrm{N}\big((\mathbf{a}\times\mathbf{b})\times\mathbf{a}\big) &= \frac{1}{16}\big((\mathbf{ab}-\mathbf{ba})\mathbf{a}-\mathbf{a}(\mathbf{ab}-\mathbf{ba})\big)\\
&\qquad\big((\mathbf{ba}-\mathbf{ab})\mathbf{a}-\mathbf{a}(\mathbf{ba}-\mathbf{ab}\big)\\
&= \frac{1}{4}\big(\mathbf{aba}-\mathbf{a}^2\mathbf{b}\big)\big(\mathbf{a}^2\mathbf{b}^2-\mathbf{aba}\big).
\end{aligned}\tag{4.0.4}$$

If $\mathbf{a}$ is time-like, we can simplify this expression to

$$\frac{1}{4}\big(\mathbf{aba}+\mathbf{b}\big)\big(-\mathbf{b}-\mathbf{aba}\big) = \frac{1}{4}\big(\underbrace{-\mathbf{a}^4\mathbf{b}^2-\mathbf{a}^2}_{=2\text{ or }0}-(\mathbf{ab})^2-(\mathbf{ba})^2\big).\tag{4.0.5}$$

If $\mathbf{a}$ is space-like, we get

$$\frac{1}{4}\big(\mathbf{aba}-\mathbf{b}\big)\big(\mathbf{b}-\mathbf{aba}\big) = \frac{1}{4}\big(\underbrace{-\mathbf{a}^4\mathbf{b}^2-\mathbf{a}^2}_{=-2\text{ or }0}+(\mathbf{ab})^2+(\mathbf{ba})^2\big).\tag{4.0.6}$$ □

4.1 Time-like solutions

Theorem 4.5. *If* $\hat{\mathbf{a}}$ *and* $\hat{\mathbf{d}}$ *are both time-like, Equation* (4.0.3) *has a 1-parameter family of time-like solutions with time-like bivector parts if* $\hat{\mathbf{a}}$ *and* $\hat{\mathbf{d}}$ *are within the same light cone (future or past).*

Proof. Assuming that the bivector part of X is time-like, we use Equation (2.5.19) to rewrite $\widehat{X}$ as $\cos\frac{\phi}{2} + \sin\frac{\phi}{2}\hat{\mathbf{x}}_+$, where $\mathrm{N}\left(\hat{\mathbf{x}}_+\right) = 1$. Substituting into (4.0.3) yields

$$\begin{aligned}\widehat{X}\hat{\mathbf{a}}\widehat{X}^\dagger = &-\sin^2\frac{\phi}{2}\left(\hat{\mathbf{x}}_+\cdot\hat{\mathbf{a}}\right)\hat{\mathbf{x}}_+ + \cos^2\frac{\phi}{2}\hat{\mathbf{a}} + \\ &2\sin\frac{\phi}{2}\cos\frac{\phi}{2}\left(\hat{\mathbf{x}}_+\times\hat{\mathbf{a}}\right) - \sin^2\frac{\phi}{2}\big(\left(\hat{\mathbf{x}}_+\times\hat{\mathbf{a}}\right)\times\hat{\mathbf{x}}_+\big) = \hat{\mathbf{d}}.\end{aligned} \tag{4.1.1}$$

By Theorem 2.30, we can write $\hat{\mathbf{a}} = -\left(\hat{\mathbf{x}}_+\cdot\hat{\mathbf{a}}\right)\hat{\mathbf{x}}_+ + \left(\hat{\mathbf{x}}_+\times\hat{\mathbf{a}}\right)\times\hat{\mathbf{x}}_+$ and $\hat{\mathbf{d}} = -(\hat{\mathbf{x}}_+\cdot\hat{\mathbf{d}})\hat{\mathbf{x}}_+ + (\hat{\mathbf{x}}_+\times\hat{\mathbf{d}})\times\hat{\mathbf{x}}_+$. This, along with applying known trigonometric identities, simplifies Equation (4.1.1) into

$$\begin{aligned}\widehat{X}\hat{\mathbf{a}}\widehat{X}^\dagger &= -\left(\hat{\mathbf{x}}_+\cdot\hat{\mathbf{a}}\right)\hat{\mathbf{x}}_+ + \sin\phi\left(\hat{\mathbf{x}}_+\times\hat{\mathbf{a}}\right) + \cos\phi\big(\left(\hat{\mathbf{x}}_+\times\hat{\mathbf{a}}\right)\times\hat{\mathbf{x}}_+\big) \\ &= -(\hat{\mathbf{x}}_+\cdot\hat{\mathbf{d}})\hat{\mathbf{x}}_+ + (\hat{\mathbf{x}}_+\times\hat{\mathbf{d}})\times\hat{\mathbf{x}}_+.\end{aligned} \tag{4.1.2}$$

Observe that both sides are now decomposed into components parallel and orthogonal to $\hat{\mathbf{x}}_+$. If we examine the parallel components, we get that

$$\hat{\mathbf{x}}_+\cdot(\hat{\mathbf{d}}-\hat{\mathbf{a}}) = 0. \tag{4.1.3}$$

In words, $\hat{\mathbf{x}}_+$ is orthogonal to vector $\hat{\mathbf{d}}-\hat{\mathbf{a}}$. Firstly note that $\hat{\mathbf{d}}-\hat{\mathbf{a}}$ can never be light-like or time-like (if we imagine a plane spanned by $\hat{\mathbf{a}}$ and $\hat{\mathbf{d}}$ intersecting the positive unit hyperboloid forming a hyperbola, $\hat{\mathbf{d}}-\hat{\mathbf{a}}$ can never be parallel to a light-like asymptote of such hyperbola). It is easily verified that any vector orthogonal to $\hat{\mathbf{d}}-\hat{\mathbf{a}}$ lies in the plane spanned by $\hat{\mathbf{a}}+\hat{\mathbf{d}}$ and $\hat{\mathbf{d}}\times\hat{\mathbf{a}}$, whose intersection with the positive unit hyperboloid forms a hyperbola. Note that $\hat{\mathbf{x}}_+$ and $-\hat{\mathbf{x}}_+$ give the exact same solution as the decomposition into parallel and orthogonal parts is unique and depends only on the Clifford norm. This also shows the necessity of $\hat{\mathbf{a}}$ and $\hat{\mathbf{d}}$ being within the same light-cone, for otherwise $\hat{\mathbf{d}}-\hat{\mathbf{a}}$ would be time-like, in which case there exists no orthogonal time-like vector $\hat{\mathbf{x}}_+$.

Since $\hat{\mathbf{a}} + \hat{\mathbf{d}}$ and $\hat{\mathbf{d}} \times \hat{\mathbf{a}}$ are orthogonal, this hyperbola can be described by the parametric equation

$$\hat{\mathbf{x}}_{+}(\theta) = \cosh\theta \frac{(\hat{\mathbf{a}} + \hat{\mathbf{d}})}{\sqrt{\mathrm{N}(\hat{\mathbf{a}} + \hat{\mathbf{d}})}} + \sinh\theta \frac{(\hat{\mathbf{d}} \times \hat{\mathbf{a}})}{\sqrt{|\mathrm{N}(\hat{\mathbf{d}} \times \hat{\mathbf{a}})|}}, \qquad \theta \in \mathbb{R}. \tag{4.1.4}$$

We conclude the proof by observing that by Lemmas 4.3, 4.4 and Corollary 4.1, the vectors $\hat{\mathbf{x}}_{+} \times \hat{\mathbf{a}}$, $(\hat{\mathbf{x}}_{+} \times \hat{\mathbf{a}}) \times \hat{\mathbf{x}}_{+}$ and $(\hat{\mathbf{x}}_{+} \times \hat{\mathbf{d}}) \times \hat{\mathbf{x}}_{+}$ in Equation (4.1.2) are all of equal magnitude and share a common plane orthogonal to $\hat{\mathbf{x}}_{+}$. Furthermore, $\hat{\mathbf{x}}_{+} \times \hat{\mathbf{a}}$ and $(\hat{\mathbf{x}}_{+} \times \hat{\mathbf{a}}) \times \hat{\mathbf{x}}_{+}$ are also mutually orthogonal, and by the Pythagorean Theorem and 2.31 there exists a circular angle $\phi \in (-\pi, \pi]$ such that

$$\sin\phi\,(\hat{\mathbf{x}}_{+} \times \hat{\mathbf{a}}) + \cos\phi\big((\hat{\mathbf{x}}_{+} \times \hat{\mathbf{a}}) \times \hat{\mathbf{x}}_{+}\big) = (\hat{\mathbf{x}}_{+} \times \hat{\mathbf{d}}) \times \hat{\mathbf{x}}, \tag{4.1.5}$$

which concludes the proof. □

Theorem 4.6. *If* $\hat{\mathbf{a}}$ *and* $\hat{\mathbf{d}}$ *are space-like vectors, Equation* (4.0.3) *has a 1-parameter family of time-like solutions with time-like bivector parts if and only if* $\hat{\mathbf{d}} - \hat{\mathbf{a}}$ *is also a space-like vector.*

Proof. The reasoning is identical to Theorem 4.5 up to Equation (4.1.3). The set of positive unit vectors orthogonal to $\hat{\mathbf{d}} - \hat{\mathbf{a}}$ can be described by a parametric equation

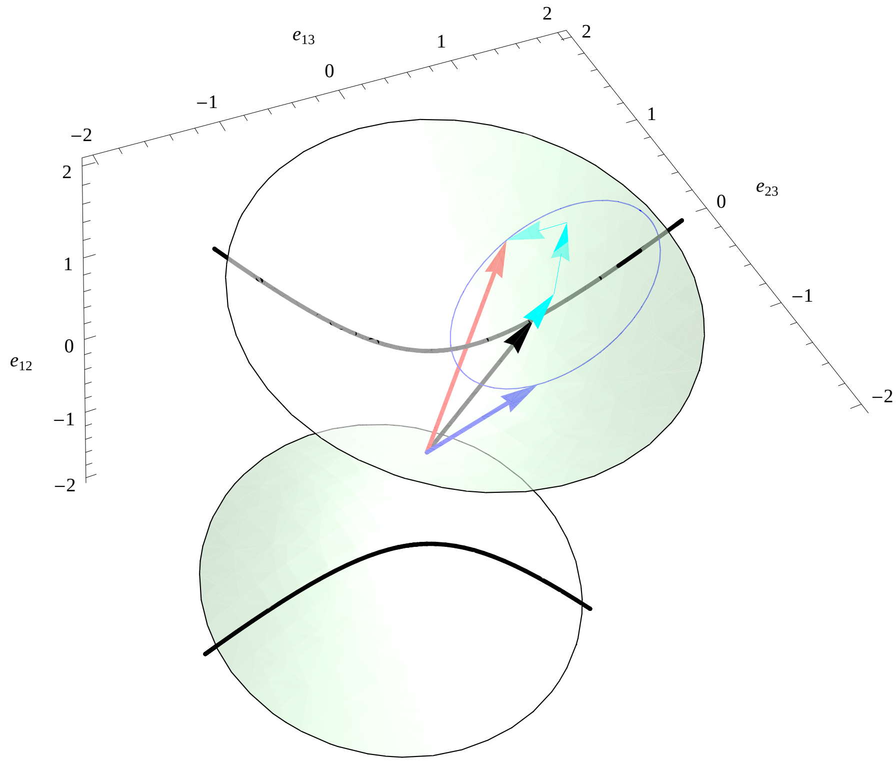
$$\hat{\mathbf{x}}_{++}(\theta) = \cosh\theta \frac{(\hat{\mathbf{d}} \times \hat{\mathbf{a}})}{\sqrt{\mathrm{N}(\hat{\mathbf{d}} \times \hat{\mathbf{a}})}} + \sinh\theta \frac{(\hat{\mathbf{a}} + \hat{\mathbf{d}})}{\sqrt{|\mathrm{N}(\hat{\mathbf{a}} + \hat{\mathbf{d}})|}}, \qquad \theta \in \mathbb{R}, \tag{4.1.6}$$

if the vector $\hat{\mathbf{d}} \times \hat{\mathbf{a}}$ is time-like and

$$\hat{\mathbf{x}}_{+-}(\theta) = \cosh\theta \frac{(\hat{\mathbf{a}} + \hat{\mathbf{d}})}{\sqrt{\mathrm{N}(\hat{\mathbf{a}} + \hat{\mathbf{d}})}} + \sinh\theta \frac{(\hat{\mathbf{d}} \times \hat{\mathbf{a}})}{\sqrt{\left|\mathrm{N}(\hat{\mathbf{d}} \times \hat{\mathbf{a}})\right|}}, \qquad \theta \in \mathbb{R}, \tag{4.1.7}$$

if $\hat{\mathbf{d}} \times \hat{\mathbf{a}}$ is space-like. In the case when $\hat{\mathbf{a}}$ and $\hat{\mathbf{d}}$ share a common plane with some light-like vector, then the vector $\hat{\mathbf{d}} \times \hat{\mathbf{a}}$ is also light-like (since $\hat{\mathbf{a}}$ and $\hat{\mathbf{d}}$ are both orthogonal to a common light-like vector), and we therefore need to construct the hyperbola of solutions differently. Using the approach from [23], let $\mathbf{u} = \frac{\hat{\mathbf{d}} - \hat{\mathbf{a}}}{\sqrt{\left|\mathrm{N}(\hat{\mathbf{d}} - \hat{\mathbf{a}})\right|}}$, and let us choose any time-like vector, say $\mathbf{e}_{12} = (1, 0, 0)$. If $\mathbf{u} \cdot \mathbf{e}_{12} = 0$, let $\mathbf{v} = \mathbf{e}_{12}$, otherwise, $\mathbf{v} = \mathbf{u} - \frac{\mathbf{e}_{12}}{\mathbf{u} \cdot \mathbf{e}_{12}}$. It

Figure 4.1: An example of usage of Theorem 4.5 with vectors $\hat{\mathbf{a}}$ in blue and $\hat{\mathbf{d}}$ in red. The set of bivector parts of all solutions is the thick black hyperbola. The thin blue ellipse is formed by the results of the rotation for fixed $\hat{\mathbf{x}}_+$ and varying circular angle ϕ.



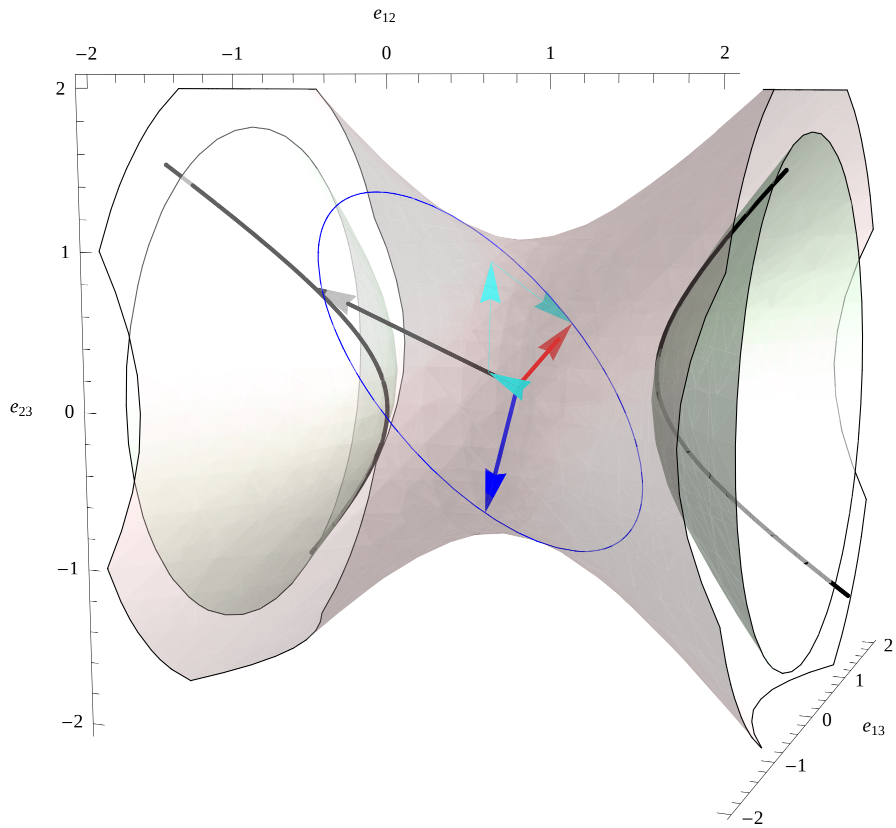
can be easily verified that $\mathbf{v}$ is both time-like and orthogonal to $\hat{\mathbf{d}} - \hat{\mathbf{a}}$. Finally, letting $\mathbf{w} = \mathbf{u} \times \mathbf{v}$, we can write the positive unit vectors orthogonal to $\hat{\mathbf{d}} - \hat{\mathbf{a}}$ as

$$\hat{\mathbf{x}}_{+0}(\theta) = \cosh\theta \frac{\mathbf{v}}{\sqrt{\mathrm{N}(\mathbf{v})}} + \sinh\theta \frac{\mathbf{w}}{\sqrt{|\mathrm{N}(\mathbf{w})|}}, \qquad \theta \in \mathbb{R}. \tag{4.1.8}$$

Observe here that $\mathrm{N}(\hat{\mathbf{d}} - \hat{\mathbf{a}}) \geq 0$ implies no time-like vector orthogonal to $\hat{\mathbf{d}} - \hat{\mathbf{a}}$ exists.

Finally, since by Lemmas 4.3 and 4.4 we know $\mathrm{N}(\hat{\mathbf{x}}_+ \times \hat{\mathbf{a}}) = \mathrm{N}\big((\hat{\mathbf{x}}_+ \times \hat{\mathbf{a}}) \times \hat{\mathbf{x}}_+\big)$ and $\mathrm{N}(\hat{\mathbf{x}}_+ \times \hat{\mathbf{a}}) = \mathrm{N}(\hat{\mathbf{x}}_+ \times \hat{\mathbf{d}})$, and by Pythagorean Theorem there exists a circular angle $\phi \in (-\pi, \pi]$ such that Equation (4.1.5) is satisfied for space-like $\hat{\mathbf{a}}$ and $\hat{\mathbf{d}}$ as well. $\square$

Figure 4.2: An example of usage of Theorem 4.6 with vectors $\hat{\mathbf{a}}$ in blue and $\hat{\mathbf{d}}$ in red. The set of bivector parts of all solutions is the thick black hyperbola. The thin blue ellipse is formed by the results of the rotation for a fixed $\hat{\mathbf{x}}_+$ and varying circular angle ϕ.



Theorem 4.7. *If* $\hat{\mathbf{a}}$ *and* $\hat{\mathbf{d}}$ *are both time-like, Equation* (4.0.3) *has a 1-parameter family of time-like solutions with space-like bivector parts, which form*

1. *A hyperbola if* $\hat{\mathbf{a}}$ *and* $\hat{\mathbf{d}}$ *are within the same light cone.*

2. *An ellipse if* $\hat{\mathbf{a}}$ *and* $\hat{\mathbf{d}}$ *are not within the same light cone.*

Proof of 1. Assume that the bivector part of $\widehat{X}$ in Equation (4.0.3) is space-like. According to Equation (2.5.19), we rewrite $\widehat{X}$ as $\cosh\frac{\psi}{2} + \sinh\frac{\psi}{2}\hat{\mathbf{x}}_-$, where $\mathrm{N}(\hat{\mathbf{x}}_-) = -1$.

Substituting into Equation (4.0.3) and expanding yields

$$\begin{aligned}\widehat{X}\hat{\mathbf{a}}\widehat{X}^{\dagger} = &-\sinh^2\frac{\psi}{2}(\hat{\mathbf{x}}_{-}\cdot\hat{\mathbf{a}})\hat{\mathbf{x}}_{-} + \cosh^2\frac{\psi}{2}\hat{\mathbf{a}} + \\ &2\cosh\frac{\psi}{2}\sinh\frac{\psi}{2}(\hat{\mathbf{x}}_{-}\times\hat{\mathbf{a}}) - \sinh^2\frac{\psi}{2}\big((\hat{\mathbf{x}}_{-}\times\hat{\mathbf{a}})\times\hat{\mathbf{x}}_{-}\big) = \hat{\mathbf{d}}.\end{aligned} \tag{4.1.9}$$

By Theorem 2.30, we can write $\hat{\mathbf{a}}$ as $(\hat{\mathbf{x}}_{-}\cdot\hat{\mathbf{a}})\hat{\mathbf{x}}_{-} - (\hat{\mathbf{x}}_{-}\times\hat{\mathbf{a}})\times\hat{\mathbf{x}}_{-}$ and $\hat{\mathbf{d}}$ as $(\hat{\mathbf{x}}_{-}\cdot\hat{\mathbf{d}})\hat{\mathbf{x}}_{-} - (\hat{\mathbf{x}}_{-}\times\hat{\mathbf{d}})\times\hat{\mathbf{x}}_{-}$. This, along with applying known hyperbolometric identities, simplifies Equation (4.1.9) to

$$\begin{aligned}\widehat{X}\hat{\mathbf{a}}\widehat{X}^{\dagger} &= (\hat{\mathbf{x}}_{-}\cdot\hat{\mathbf{a}})\hat{\mathbf{x}}_{-} + \sinh\psi(\hat{\mathbf{x}}_{-}\times\hat{\mathbf{a}}) - \cosh\psi\big((\hat{\mathbf{x}}_{-}\times\hat{\mathbf{a}})\times\hat{\mathbf{x}}_{-}\big) \\ &= (\hat{\mathbf{x}}_{-}\cdot\hat{\mathbf{d}})\hat{\mathbf{x}}_{-} - (\hat{\mathbf{x}}_{-}\times\hat{\mathbf{d}})\times\hat{\mathbf{x}}_{-}.\end{aligned} \tag{4.1.10}$$

Assume now that $\hat{\mathbf{a}}$ and $\hat{\mathbf{d}}$ are within the same light cone. Then the $\cosh\psi$ in this equation is positive and thus the scaled component of $\hat{\mathbf{a}}$ perpendicular to $\hat{\mathbf{x}}$ remains within the same light cone. As with (4.1.2), we can easily see that the parallel components are equal if $\hat{\mathbf{x}}_{-}$ is orthogonal to $(\hat{\mathbf{d}}-\hat{\mathbf{a}})$. This is satisfied by any vector from the plane spanned by vectors $\hat{\mathbf{a}}+\hat{\mathbf{d}}$ and $\hat{\mathbf{d}}\times\hat{\mathbf{a}}$. As $\hat{\mathbf{a}}+\hat{\mathbf{d}}$ is time-like, the intersection of the orthogonal plane with the negative unit hyperboloid forms a hyperbola described by the equation

$$\hat{\mathbf{x}}_{-}(\theta) = \cosh\theta\frac{(\hat{\mathbf{d}}\times\hat{\mathbf{a}})}{\sqrt{|\mathrm{N}(\hat{\mathbf{d}}\times\hat{\mathbf{a}})|}} + \sinh\theta\frac{(\hat{\mathbf{a}}+\hat{\mathbf{d}})}{\sqrt{\mathrm{N}(\hat{\mathbf{a}}+\hat{\mathbf{d}})}}, \qquad \theta\in\mathbb{R}. \tag{4.1.11}$$

By Lemma 4.4, Corollary 4.1 and the Pythagorean theorem, there exists a hyperbolic angle $\psi\in\mathbb{R}$ such that

$$\sinh\psi(\hat{\mathbf{x}}_{-}\times\hat{\mathbf{a}}) - \cosh\psi\big((\hat{\mathbf{x}}_{-}\times\hat{\mathbf{a}})\times\hat{\mathbf{x}}_{-}\big) = -(\hat{\mathbf{x}}_{-}\times\hat{\mathbf{d}})\times\hat{\mathbf{x}}_{-}. \tag{4.1.12}$$

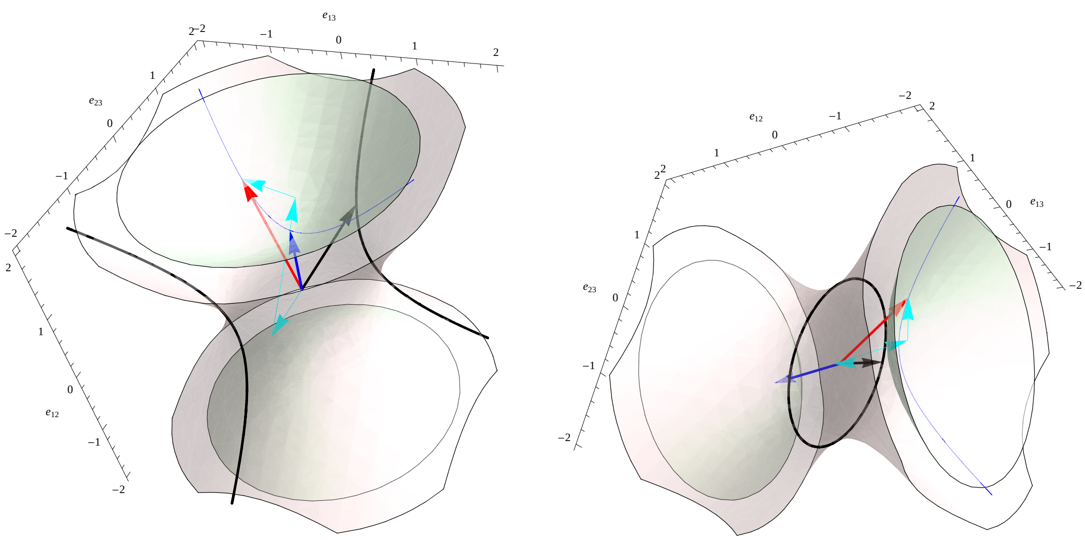
□

Proof of 2. Returning now to Equation (4.1.10), assume that $\hat{\mathbf{a}}$ and $\hat{\mathbf{d}}$ are not within the same light cone but $\hat{\mathbf{a}}\neq-\hat{\mathbf{d}}$. Then $\cosh\psi$ is negative and thus the component of $\hat{\mathbf{a}}$ perpendicular to $\hat{\mathbf{x}}$ is scaled and reflected. Furthermore, because $\hat{\mathbf{d}}-\hat{\mathbf{a}}$ is time-like and thus the plane orthogonal to $\hat{\mathbf{d}}-\hat{\mathbf{a}}$ is space-like, its intersection with the negative unit hyperboloid forms an ellipse described by the equation

$$\hat{\mathbf{x}}_{-}(\theta) = \cos\theta\frac{(\hat{\mathbf{a}}+\hat{\mathbf{d}})}{\sqrt{|\mathrm{N}(\hat{\mathbf{a}}+\hat{\mathbf{d}})|}} + \sin\theta\frac{(\hat{\mathbf{d}}\times\hat{\mathbf{a}})}{\sqrt{|\mathrm{N}(\hat{\mathbf{d}}\times\hat{\mathbf{a}})|}}, \qquad \theta\in(-\pi,\pi]. \tag{4.1.13}$$

In the special case $\hat{\mathbf{a}} = -\hat{\mathbf{d}}$, we replace the zero vectors $\hat{\mathbf{a}}+\hat{\mathbf{d}}$ and $\hat{\mathbf{d}}\times\hat{\mathbf{a}}$ with normalized vectors in the direction of $\hat{\mathbf{a}}\times\mathbf{e}_{13}$ and $(\hat{\mathbf{a}}\times\mathbf{e}_{13})\times\hat{\mathbf{a}}$. In the case $\hat{\mathbf{a}} = -\hat{\mathbf{d}} = \pm\mathbf{e}_{1,3}$, we use vectors $\mathbf{e}_{13}$ and $\mathbf{e}_{23}$. The rest of the proof is identical to the case with $\hat{\mathbf{a}}$ and $\hat{\mathbf{d}}$ being within the same light cone. □

Figure 4.3: An example of usage of Theorem 4.7 with vectors $\hat{\mathbf{a}}$ in blue and $\hat{\mathbf{d}}$ in red. The set of bivector parts of all solutions is the thick black curve. The thin blue curve is formed by the results of the rotation for a fixed $\hat{\mathbf{x}}_-$ and varying hyperbolic angle ψ.



(a) $\hat{\mathbf{a}}$ and $\hat{\mathbf{d}}$ are within the same light cone.

(b) $\hat{\mathbf{a}}$ and $\hat{\mathbf{d}}$ are not within the same light cone.

Theorem 4.8. *If $\hat{\mathbf{a}}$ and $\hat{\mathbf{d}}$ are both space-like, Equation* (4.0.3) *has a 1-parameter family of time-like solutions with space-like bivector parts. We distinguish three cases:*

1. *A hyperbola with four or two points excluded if $\hat{\mathbf{d}}-\hat{\mathbf{a}}$ is space-like.*

2. *An ellipse with four points excluded if $\hat{\mathbf{d}}-\hat{\mathbf{a}}$ is time-like.*

3. *A pair of parallel lines with two points excluded if $\hat{\mathbf{d}}-\hat{\mathbf{a}}$ is light-like.*

Proof of 1. The reasoning is identical to the previous Theorem 4.7 up to Equation (4.1.10). If $\hat{\mathbf{d}}-\hat{\mathbf{a}}$ is space-like, then the set of negative unit vectors orthogonal to $\hat{\mathbf{d}}-\hat{\mathbf{a}}$

can be described by the parametric equation

$$\hat{\mathbf{x}}_{-+}(\theta) = \cosh\theta \frac{(\hat{\mathbf{a}} + \hat{\mathbf{d}})}{\sqrt{|\mathrm{N}(\hat{\mathbf{a}} + \hat{\mathbf{d}})|}} + \sinh\theta \frac{(\hat{\mathbf{d}} \times \hat{\mathbf{a}})}{\sqrt{\mathrm{N}(\hat{\mathbf{d}} \times \hat{\mathbf{a}})}}, \qquad \theta \in \mathbb{R}, \tag{4.1.14}$$

if $\hat{\mathbf{d}} \times \hat{\mathbf{a}}$ is time-like and

$$\hat{\mathbf{x}}_{--}(\theta) = \cosh\theta \frac{(\hat{\mathbf{d}} \times \hat{\mathbf{a}})}{\sqrt{|\mathrm{N}(\hat{\mathbf{d}} \times \hat{\mathbf{a}})|}} + \sinh\theta \frac{(\hat{\mathbf{a}} + \hat{\mathbf{d}})}{\sqrt{\mathrm{N}(\hat{\mathbf{a}} + \hat{\mathbf{d}})}}, \qquad \theta \in \mathbb{R}, \tag{4.1.15}$$

if $\hat{\mathbf{d}} \times \hat{\mathbf{a}}$ is space-like. In the case when $\hat{\mathbf{d}} \times \hat{\mathbf{a}}$ is light-like, we use the same approach as in Theorem 4.6, Equation (4.1.8). Note that this also covers the case of $\hat{\mathbf{d}} = -\hat{\mathbf{a}}$.

By Lemma 4.4 and the Pythagorean Theorem there exists a hyperbolic angle $\psi \in \mathbb{R}$ such that Equation (4.1.12) is satisfied for space-like $\hat{\mathbf{a}}$ and $\hat{\mathbf{d}}$ as well, with the exception of four points, for which

$$\left|\frac{\mathrm{N}(\hat{\mathbf{d}} \times \hat{\mathbf{a}})}{\mathrm{N}(\hat{\mathbf{a}} + \hat{\mathbf{d}})}\right| = \begin{cases} \tanh^2\theta & \text{if } \hat{\mathbf{x}}_- = \hat{\mathbf{x}}_{-+}, \\ \coth^2\theta & \text{if } \hat{\mathbf{x}}_- = \hat{\mathbf{x}}_{--}. \end{cases} \tag{4.1.16}$$

By expanding $\mathrm{N}(\hat{\mathbf{x}}_- \times \hat{\mathbf{a}})$, we can see that in these cases

$$\mathrm{N}(\hat{\mathbf{x}}_- \times \hat{\mathbf{a}}) = \mathrm{N}\big((\hat{\mathbf{x}}_- \times \hat{\mathbf{a}}) \times \hat{\mathbf{x}}_-\big) = \mathrm{N}(\hat{\mathbf{x}}_- \times \hat{\mathbf{d}}) = \mathrm{N}\big((\hat{\mathbf{x}}_- \times \hat{\mathbf{d}}) \times \hat{\mathbf{x}}_-\big) = 0,$$

which means that $\hat{\mathbf{x}}_- \times \hat{\mathbf{a}}$ is a multiple of $(\hat{\mathbf{x}}_- \times \hat{\mathbf{a}}) \times \hat{\mathbf{x}}_-$ but has a distinct direction from $(\hat{\mathbf{x}}_- \times \hat{\mathbf{d}}) \times \hat{\mathbf{x}}_-$ since $\mathrm{N}(\hat{\mathbf{a}} \times \hat{\mathbf{d}}) \neq 0$.

We can easily show that the expression $\left|\frac{\mathrm{N}(\hat{\mathbf{d}} \times \hat{\mathbf{a}})}{\mathrm{N}(\hat{\mathbf{a}} + \hat{\mathbf{d}})}\right|$ is always within the range of the $\tanh^2\theta$ or $\coth^2\theta$ functions. For example if $\hat{\mathbf{d}} \times \hat{\mathbf{a}}$ is time-like, then from Lemmas 4.2 and 4.3 we have

$$\begin{aligned} \mathrm{N}(\hat{\mathbf{a}} + \hat{\mathbf{d}}) &= -2 - 2(\hat{\mathbf{a}} \cdot \hat{\mathbf{d}}) \Leftrightarrow \hat{\mathbf{a}} \cdot \hat{\mathbf{d}} > -1, \\ \mathrm{N}(\hat{\mathbf{a}} + \hat{\mathbf{d}}) &= -2 + 2(\hat{\mathbf{a}} \cdot \hat{\mathbf{d}}) \Leftrightarrow \hat{\mathbf{a}} \cdot \hat{\mathbf{d}} < 1, \end{aligned}$$

and therefore

$$0 < \left|\frac{\mathrm{N}(\hat{\mathbf{d}} \times \hat{\mathbf{a}})}{\mathrm{N}(\hat{\mathbf{a}} + \hat{\mathbf{d}})}\right| = \frac{1 - (\hat{\mathbf{a}} \cdot \hat{\mathbf{d}})^2}{2(1 + \hat{\mathbf{a}} \cdot \hat{\mathbf{d}})} = \frac{1 - \hat{\mathbf{a}} \cdot \hat{\mathbf{d}}}{2} < 1,$$

which is the range of $\tanh^2\theta$. These singular cases are caused by $\hat{\mathbf{x}}_-$ and $\hat{\mathbf{a}}$ sharing a common plane with a light-like vector. This also occurs when $\hat{\mathbf{d}} \times \hat{\mathbf{a}}$ is light-like; using notation from Equation (4.1.8), we can describe this case by equation

$$\mathrm{N}\big(\hat{\mathbf{x}}_{-0}(\theta) \times \hat{\mathbf{a}}\big) = 0, \tag{4.1.17}$$

however, it is much more difficult to identify the corresponding θ values in a closed form such as in Equation (4.1.16). Nevertheless, we know that only two such values exist since $\hat{\mathbf{x}}_{-0}$ never meets with the plane containing $\hat{\mathbf{d}} \times \hat{\mathbf{a}}$ as a result of the construction we used. □

Proof of 2. If $\hat{\mathbf{d}} - \hat{\mathbf{a}}$ is time-like, then set of negative unit vectors orthogonal to $\hat{\mathbf{d}} - \hat{\mathbf{a}}$ can be described by a parametric equation

$$\hat{\mathbf{x}}_-(\theta) = \cos\theta \frac{(\hat{\mathbf{a}} + \hat{\mathbf{d}})}{\sqrt{|\mathrm{N}(\hat{\mathbf{a}} + \hat{\mathbf{d}})|}} + \sin\theta \frac{(\hat{\mathbf{d}} \times \hat{\mathbf{a}})}{\sqrt{\left|\mathrm{N}(\hat{\mathbf{d}} \times \hat{\mathbf{a}})\right|}}, \qquad \theta \in (-\pi, \pi]. \tag{4.1.18}$$

Observe that in this case, $\hat{\mathbf{d}} \times \hat{\mathbf{a}}$ is always space-like. The rest of the proof is analogous, with the four vectors excluded from the solution set satisfying

$$\left|\frac{\mathrm{N}(\hat{\mathbf{d}} \times \hat{\mathbf{a}})}{\mathrm{N}(\hat{\mathbf{a}} + \hat{\mathbf{d}})}\right| = \tan^2\theta. \tag{4.1.19}$$

□

Proof of 3. If $\hat{\mathbf{d}} - \hat{\mathbf{a}}$ is light-like, then the plane orthogonal to $\hat{\mathbf{d}} - \hat{\mathbf{a}}$ also contains $\hat{\mathbf{d}} - \hat{\mathbf{a}}$, and its intersection with the negative unit hyperboloid forms two parallel lines. Observe now that both $\hat{\mathbf{a}}$ and $\hat{\mathbf{d}}$ also lie on one of these lines. We denote by $\hat{\mathbf{x}}_{-+}$ the vectors $\hat{\mathbf{x}}_-$ lying on the same line as $\hat{\mathbf{a}}$ and $\hat{\mathbf{d}}$ and by $\hat{\mathbf{x}}_{--}$ vectors $\hat{\mathbf{x}}_-$ lying on the other line. Since $\mathrm{N}(\hat{\mathbf{x}}_- \times \hat{\mathbf{a}})$ must be 0, we can deduce from the Lemma 4.3 that $\hat{\mathbf{x}}_- \cdot \hat{\mathbf{a}} = \pm 1$ and therefore $\hat{\mathbf{a}}_\parallel$, the component of $\hat{\mathbf{a}}$ parallel to $\hat{\mathbf{x}}_-$, is

$$\hat{\mathbf{a}}_\parallel = (\hat{\mathbf{x}}_- \cdot \hat{\mathbf{a}})\hat{\mathbf{x}}_- = \begin{cases} \hat{\mathbf{x}}_- & \text{if } \hat{\mathbf{x}}_- = \hat{\mathbf{x}}_{-+}, \\ -\hat{\mathbf{x}}_- & \text{if } \hat{\mathbf{x}}_- = \hat{\mathbf{x}}_{--}. \end{cases} \tag{4.1.20}$$

This implies that $\hat{\mathbf{a}}_\perp$, the component of $\hat{\mathbf{a}}$ orthogonal to $\hat{\mathbf{x}}_-$, is

$$\hat{\mathbf{a}}_\perp = -(\hat{\mathbf{x}}_- \times \hat{\mathbf{a}}) \times \hat{\mathbf{x}}_- = \begin{cases} \hat{\mathbf{a}} - \hat{\mathbf{x}}_- & \text{if } \hat{\mathbf{x}}_- = \hat{\mathbf{x}}_{-+}, \\ \hat{\mathbf{a}} + \hat{\mathbf{x}}_- & \text{if } \hat{\mathbf{x}}_- = \hat{\mathbf{x}}_{--}. \end{cases} \tag{4.1.21}$$

Both preceding observations are valid for $\hat{\mathbf{d}}_{\parallel}$ and $\hat{\mathbf{d}}_{\perp}$ as well. Lastly, through a rather tedious computation, involving expressing both $\hat{\mathbf{x}}$ and $\hat{\mathbf{a}}$ in spherical coordinates (any light-like vector has an inclination of $\pm\frac{\pi}{4}$ in the direction of $\mathbf{e}_{12}$) and then expanding the commutator product into coordinates, we can also show that $\hat{\mathbf{x}}_{-} \times \hat{\mathbf{a}}$ is equal to $-\hat{\mathbf{a}}_{\perp}$ if $\hat{\mathbf{x}}_{-} = \hat{\mathbf{x}}_{-+}$ and $\hat{\mathbf{a}}_{\perp}$ if $\hat{\mathbf{x}}_{-} = \hat{\mathbf{x}}_{--}$. Combining these observations, substituting back into (4.1.10) and recalling that we allow the values of cosh function to be negative yields

$$\widehat{X}\hat{\mathbf{a}}\widehat{X}^{\dagger} = \begin{cases} \hat{\mathbf{x}}_{-} - \sinh\psi\hat{\mathbf{a}}_{\perp} + \cosh\psi\hat{\mathbf{a}}_{\perp} = \hat{\mathbf{x}}_{-} \pm e^{\mp\psi}\hat{\mathbf{a}}_{\perp} & \text{if } \hat{\mathbf{x}}_{-} = \hat{\mathbf{x}}_{-+}, \\ \hat{\mathbf{x}}_{-} + \sinh\psi\hat{\mathbf{a}}_{\perp} + \cosh\psi\hat{\mathbf{a}}_{\perp} = \hat{\mathbf{x}}_{-} \pm e^{\pm\psi}\hat{\mathbf{a}}_{\perp} & \text{if } \hat{\mathbf{x}}_{-} = \hat{\mathbf{x}}_{--}. \end{cases} \tag{4.1.22}$$

Thus we conclude that for any $\hat{\mathbf{x}}_{-}$, there exists a hyperbolic angle ψ such that Equation (4.1.12) is satisfied. The two exceptions are vectors $\hat{\mathbf{a}}$ and $-\hat{\mathbf{a}}$, which do not have any components orthogonal to $\hat{\mathbf{x}}_{-}$, that is themselves. Finally, note however that we cannot define the hyperbolic angle ψ in terms of $\hat{\mathbf{a}}_{\perp}$ and $\hat{\mathbf{d}}_{\perp}$ as in the previous cases, as $\mathrm{N}(\hat{\mathbf{a}}_{\perp}) = \mathrm{N}(\hat{\mathbf{d}}_{\perp}) = 0$. □

Theorem 4.9. *If $\hat{\mathbf{a}}$ and $\hat{\mathbf{d}}$ are both time-like, Equation* (4.0.3) *has four time-like solutions with light-like bivector parts if $\hat{\mathbf{a}}$ and $\hat{\mathbf{d}}$ are within the same light cone.*

Proof. Assume that the bivector part of $\widehat{X}$ in Equation (4.0.3) is light-like; then by Equation (2.5.19) $\widehat{X} = \pm 1 + \mathbf{x}$ for some light-like vector $\mathbf{x}$. Expanding Equation (4.0.3) then yields

$$\widehat{X}\hat{\mathbf{a}}\widehat{X}^{\dagger} = -(\mathbf{x}\cdot\hat{\mathbf{a}})\mathbf{x} + \hat{\mathbf{a}} \pm \mathbf{x}\times\hat{\mathbf{a}} \mp \hat{\mathbf{a}}\times\mathbf{x} - (\mathbf{x}\times\hat{\mathbf{a}})\times\mathbf{x} = \hat{\mathbf{d}}. \tag{4.1.23}$$

From Theorem 2.30, we can see that for a light-like vector $\mathbf{x}$, we have $(\mathbf{x}\cdot\hat{\mathbf{a}})\mathbf{x} = (\mathbf{x}\times\hat{\mathbf{a}})\times\mathbf{x}$. This simplifies Equation (4.1.23) to

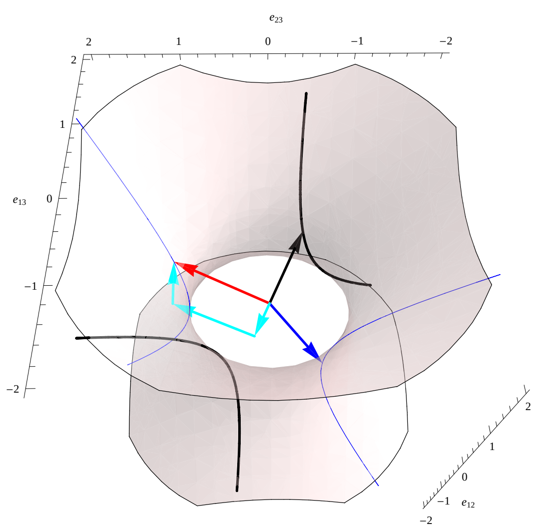
$$\widehat{X}\hat{\mathbf{a}}\widehat{X}^{\dagger} = \hat{\mathbf{a}} - 2(\mathbf{x}\cdot\hat{\mathbf{a}})\mathbf{x} \pm 2\mathbf{x}\times\hat{\mathbf{a}} = \hat{\mathbf{d}}, \tag{4.1.24}$$

hence

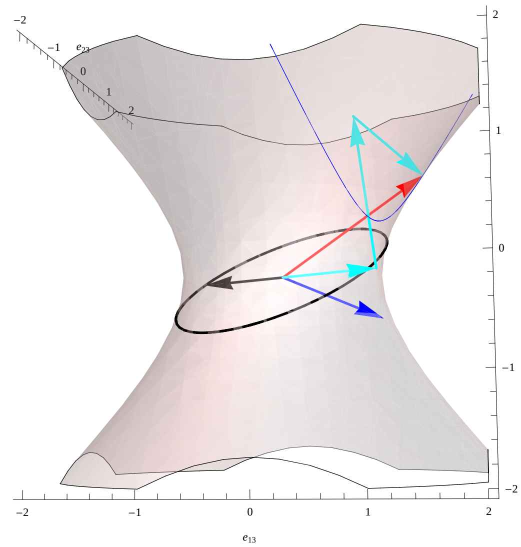
$$-2(\mathbf{x}\cdot\hat{\mathbf{a}})\mathbf{x} \pm 2\mathbf{x}\times\hat{\mathbf{a}} = \hat{\mathbf{d}} - \hat{\mathbf{a}}. \tag{4.1.25}$$

Recall that a plane orthogonal to a given light-like vector is tangent to the light cone and contains the said vector. This identifies the direction of $\mathbf{x}$ as orthogonal to the

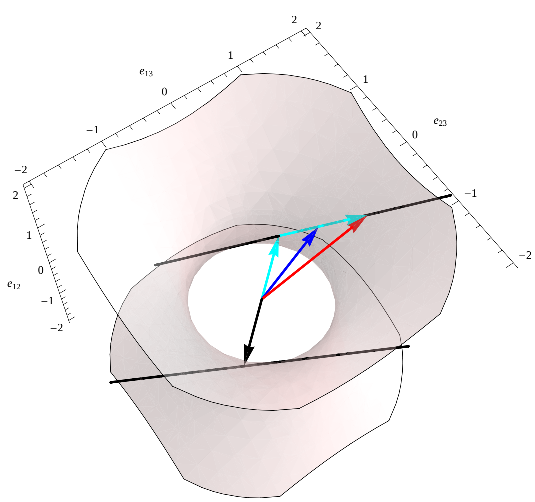
Figure 4.4: An example of usage of Theorem 4.8 with vectors $\hat{\mathbf{a}}$ in blue and $\hat{\mathbf{d}}$ in red. The set of bivector parts of all solutions is the thick black curve. The thin blue curve is formed by the results of the rotation for a fixed $\hat{\mathbf{x}}_-$ and varying hyperbolic angle ψ.



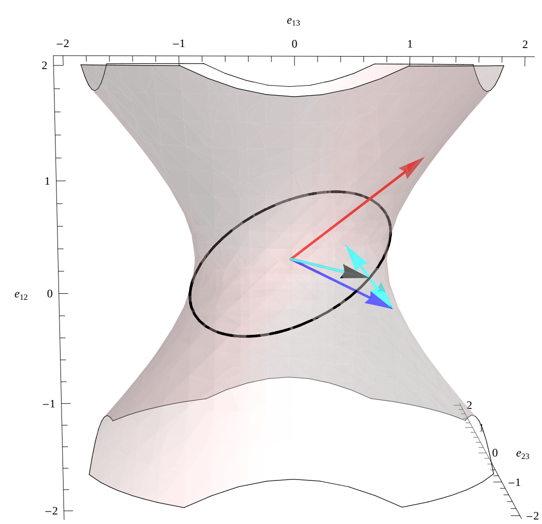
(a) $\hat{\mathbf{d}} - \hat{\mathbf{a}}$ is space-like.



(b) $\hat{\mathbf{d}} - \hat{\mathbf{a}}$ is time-like.



(c) $\hat{\mathbf{d}} - \hat{\mathbf{a}}$ is light-like.



(d) The singular case; $\hat{\mathbf{x}}_- \times \hat{\mathbf{a}}$ is parallel to $(\hat{\mathbf{x}}_- \times \hat{\mathbf{a}}) \times \hat{\mathbf{x}}_-$.

space-like vector $\hat{\mathbf{d}} - \hat{\mathbf{a}}$. It is easy to see that any light-like vector with this direction must be a multiple of

$$\mathbf{x}_{0\pm} = \frac{\hat{\mathbf{a}} + \hat{\mathbf{d}}}{\sqrt{\mathrm{N}(\hat{\mathbf{a}} + \hat{\mathbf{d}})}} \mp \frac{\hat{\mathbf{a}} \times \hat{\mathbf{d}}}{\sqrt{\left|\mathrm{N}(\hat{\mathbf{a}} \times \hat{\mathbf{d}})\right|}}. \tag{4.1.26}$$

This also shows the necessity of $\hat{\mathbf{a}}$ and $\hat{\mathbf{d}}$ being within the same light cone; otherwise $\hat{\mathbf{d}} - \hat{\mathbf{a}}$ would be time-like and have no light-like vector orthogonal to it. We now need to find a scale coefficient $\alpha \in \mathbb{R}$ of $\mathbf{x}_{0\pm}$ such that (4.1.25) is satisfied. As $(\mathbf{x}_{0\pm} \cdot \hat{\mathbf{a}})\mathbf{x}_{0\pm}$ and $\mathbf{x}_{0\pm} \times \hat{\mathbf{a}}$ are orthogonal, we get from Pythagorean Theorem that

$$\mathrm{N}\big(-2(\alpha\mathbf{x}_{0\pm} \cdot \hat{\mathbf{a}})\alpha\mathbf{x}_{0\pm} \pm 2\alpha\mathbf{x}_{0\pm} \times \hat{\mathbf{a}}\big) = \mathrm{N}(2\alpha\mathbf{x}_{0\pm} \times \hat{\mathbf{a}}) = \mathrm{N}(\hat{\mathbf{d}} - \hat{\mathbf{a}}). \tag{4.1.27}$$

Using the notation $n_+ := \mathrm{N}(\hat{\mathbf{a}} + \hat{\mathbf{d}})$ and $n_\times := \mathrm{N}(\hat{\mathbf{a}} \times \hat{\mathbf{d}})$, expanding the left hand side of Equation (4.1.27) yields

$$\mathrm{N}(2\alpha\mathbf{x}_{0\pm} \times \hat{\mathbf{a}}) = 4\alpha^2 \left(\frac{\mathrm{N}(\hat{\mathbf{d}} \times \hat{\mathbf{a}})}{n_+} + \frac{\mathrm{N}\big((\hat{\mathbf{a}} \times \hat{\mathbf{d}}) \times \hat{\mathbf{a}}\big)}{|n_\times|} \right). \tag{4.1.28}$$

Recalling Lemma 4.4, we have that $\mathrm{N}(\hat{\mathbf{d}} \times \hat{\mathbf{a}}) = \mathrm{N}\big((\hat{\mathbf{a}} \times \hat{\mathbf{d}}) \times \hat{\mathbf{a}}\big)$. Applying Lemmas 4.2 and 4.3 we get

$$\begin{aligned} \mathrm{N}(2\alpha\mathbf{x}_{0\pm} \times \hat{\mathbf{a}}) &= 4\alpha^2 \left(\frac{(1 - \hat{\mathbf{a}} \cdot \hat{\mathbf{d}})(1 + \hat{\mathbf{a}} \cdot \hat{\mathbf{d}})}{2(1 - \hat{\mathbf{a}} \cdot \hat{\mathbf{d}})} - 1 \right) = 2\alpha^2(-1 + \hat{\mathbf{a}} \cdot \hat{\mathbf{d}}) \\ &= -\alpha^2 \mathrm{N}(\hat{\mathbf{d}} + \hat{\mathbf{a}}), \end{aligned} \tag{4.1.29}$$

thus, substituting back into Equation (4.1.27) yields

$$\alpha_\pm = \pm\sqrt{\left| \frac{\mathrm{N}(\hat{\mathbf{d}} - \hat{\mathbf{a}})}{\mathrm{N}(\hat{\mathbf{d}} + \hat{\mathbf{a}})} \right|}. \tag{4.1.30}$$

The four solutions then are

$$X_{1,2} = [1, \alpha_\pm \mathbf{x}_{0\pm}], \qquad X_{3,4} = [-1, \alpha_\pm \mathbf{x}_{0\mp}].$$

For each of these solutions, we can verify that $-2(\alpha\mathbf{x}_0 \cdot \hat{\mathbf{a}})\alpha\mathbf{x}_0 \pm 2\alpha\mathbf{x}_0 \times \hat{\mathbf{a}}$ is a multiple of $\hat{\mathbf{d}} - \hat{\mathbf{a}}$. For example, for X_1, if we furthermore let $n_- := \mathrm{N}(\hat{\mathbf{d}} - \hat{\mathbf{a}})$ and recall Theorem

2.30, we have

$$
\begin{aligned}
&-2\big(\,(\alpha_+\mathbf{x}_{0+})\cdot\hat{\mathbf{a}}\big)\,(\alpha_+\mathbf{x}_{0+}) + 2\big(\,(\alpha_+\mathbf{x}_{0+})\times\hat{\mathbf{a}}\big)\\
&\quad= -2\frac{|n_-|}{n_+}\left(\frac{\overbrace{-1+\hat{\mathbf{a}}\cdot\hat{\mathbf{d}}}^{=-\frac{n_+}{2}}}{\sqrt{n_+}}\right)\left(\frac{\hat{\mathbf{a}}+\hat{\mathbf{d}}}{\sqrt{n_+}}+\frac{\hat{\mathbf{a}}\times\hat{\mathbf{d}}}{\sqrt{|n_\times|}}\right)+2\sqrt{\frac{|n_-|}{n_+}}\left(\frac{\hat{\mathbf{d}}\times\hat{\mathbf{a}}}{\sqrt{n_+}}+\frac{(\hat{\mathbf{a}}\times\hat{\mathbf{d}})\times\hat{\mathbf{a}}}{\sqrt{|n_\times|}}\right)\\
&\quad= \frac{-n_-(\hat{\mathbf{a}}+\hat{\mathbf{d}})+2\sqrt{|n_-|}(\hat{\mathbf{a}}\times\hat{\mathbf{d}})-2\sqrt{|n_-|}(\hat{\mathbf{a}}\times\hat{\mathbf{d}})+4\big((\hat{\mathbf{a}}\times\hat{\mathbf{d}})\times\hat{\mathbf{a}}\big)}{n_+}\\
&\quad= \frac{-2(1+\hat{\mathbf{a}}\cdot\hat{\mathbf{d}})(\hat{\mathbf{a}}+\hat{\mathbf{d}})+4\big((\hat{\mathbf{a}}\times\hat{\mathbf{d}})\times\hat{\mathbf{a}}\big)}{2(1-\hat{\mathbf{a}}\cdot\hat{\mathbf{d}})} = \frac{\big((\hat{\mathbf{a}}\times\hat{\mathbf{d}})\times\hat{\mathbf{a}}\big)-\big((\hat{\mathbf{d}}\times\hat{\mathbf{a}})\times\hat{\mathbf{d}}\big)}{1-\hat{\mathbf{a}}\cdot\hat{\mathbf{d}}}\\
&\quad= \frac{(\hat{\mathbf{a}}\times\hat{\mathbf{d}})\times(\hat{\mathbf{a}}+\hat{\mathbf{d}})}{1-\hat{\mathbf{a}}\cdot\hat{\mathbf{d}}}.
\end{aligned}
$$

Therefore the resulting vector is orthogonal to both $\hat{\mathbf{a}}\times\hat{\mathbf{d}}$ and $\hat{\mathbf{a}}+\hat{\mathbf{d}}$, and through a somewhat tedious computation we can show that this is indeed exactly the vector $\hat{\mathbf{d}}-\hat{\mathbf{a}}$. □

Theorem 4.10. *If* $\hat{\mathbf{a}}$ *and* $\hat{\mathbf{d}}$ *are both space-like, Equation* (4.0.3) *has finitely many time-like solutions with light-like bivector parts. In particular:*

1. *Four solutions if* $\hat{\mathbf{d}}-\hat{\mathbf{a}}$ *is space-like and* $\hat{\mathbf{a}}\times\hat{\mathbf{d}}$ *is not light-like.*
2. *Two solutions if:*
   (*a*) $\hat{\mathbf{d}}-\hat{\mathbf{a}}$ *is space-like and* $\hat{\mathbf{a}}\times\hat{\mathbf{d}}$ *is light-like.*
   (*b*) $\hat{\mathbf{d}}-\hat{\mathbf{a}}$ *is light-like.*
3. *No solutions if* $\hat{\mathbf{d}}-\hat{\mathbf{a}}$ *is time-like.*

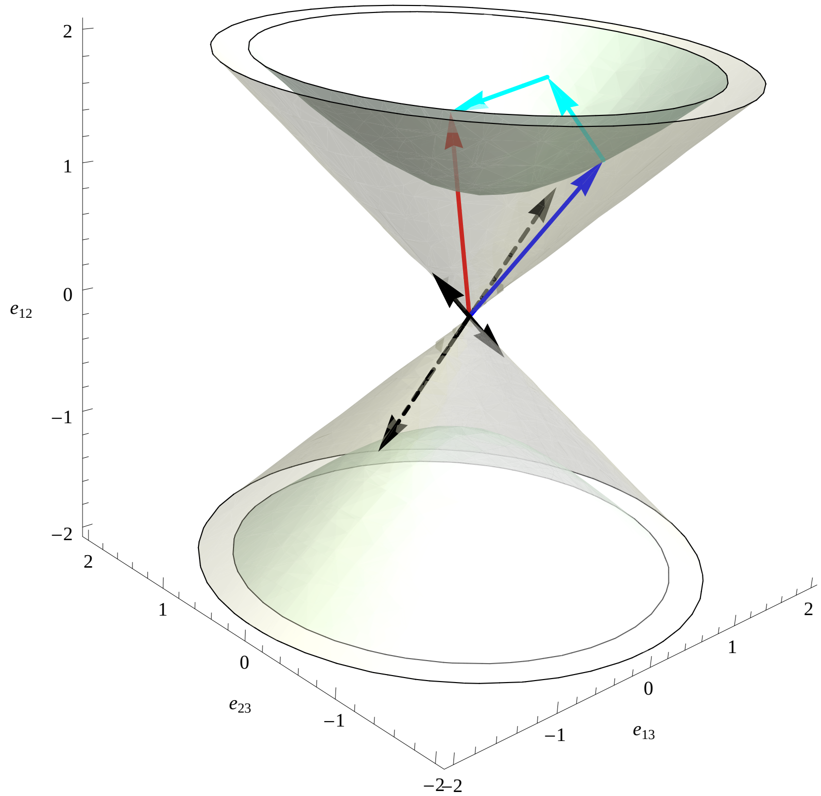
Proof of 1. If $\hat{\mathbf{d}}-\hat{\mathbf{a}}$ is space-like and $\hat{\mathbf{a}}\times\hat{\mathbf{d}}$ is not light-like, the reasoning is fully identical to Theorem 4.9, with the exception of the formula for the vector $\mathbf{x}_{0\pm}$ orthogonal to $\hat{\mathbf{d}}-\hat{\mathbf{a}}$ being

$$\mathbf{x}_{0\pm} = \frac{\hat{\mathbf{a}}+\hat{\mathbf{d}}}{\sqrt{\big|\mathrm{N}(\hat{\mathbf{a}}+\hat{\mathbf{d}})\big|}} \pm \frac{\hat{\mathbf{a}}\times\hat{\mathbf{d}}}{\sqrt{\big|\mathrm{N}(\hat{\mathbf{a}}\times\hat{\mathbf{d}})\big|}}. \tag{4.1.31}$$

The four solutions are then again

$$X_{1,2} = [1, \alpha_\pm\mathbf{x}_{0\mp}], \qquad X_{3,4} = [-1, \alpha_\pm\mathbf{x}_{0\pm}],$$

Figure 4.5: An example of usage of Theorem 4.9 with vectors $\hat{\mathbf{a}}$ in blue and $\hat{\mathbf{d}}$ in red. The set of bivector parts of all solutions are in dashed black.



where

$$\alpha_{\pm} = \pm\sqrt{\left|\frac{\mathrm{N}(\hat{\mathbf{d}} - \hat{\mathbf{a}})}{\mathrm{N}(\hat{\mathbf{d}} + \hat{\mathbf{a}})}\right|}.$$

$\square$

Proof of 2.(a). If $\hat{\mathbf{d}} - \hat{\mathbf{a}}$ is space-like and $\hat{\mathbf{a}} \times \hat{\mathbf{d}}$ is light-like, we again use the approach from Theorem 4.6 to generate components of $\mathbf{x}_{0\pm}$. Note however that one of the vectors obtained will be $\hat{\mathbf{a}} \times \hat{\mathbf{d}}$, which is not a solution, since if we substitute into Equation (4.1.25), the left hand side is a light-like vector while the right hand side is space-like by assumption. Once again, as a result of the construction used to obtain $\mathbf{x}_{0\pm}$, deriving a formula for the scaling coefficient α in this case is complicated, apart from stating that it must satisfy the equation

$$4\alpha^2 \mathrm{N}(\mathbf{x}_0 \times \hat{\mathbf{a}}) = \mathrm{N}(\hat{\mathbf{d}} - \hat{\mathbf{a}}). \tag{4.1.32}$$

□

Proof of 2.(b). If $\hat{\mathbf{d}} - \hat{\mathbf{a}}$ is light-like, then the two solutions clearly are

$$X_{1,2} = \left[\pm 1, \mp\frac{1}{2}(\hat{\mathbf{d}} - \hat{\mathbf{a}})\right],$$

since $(\hat{\mathbf{d}} - \hat{\mathbf{a}}) \cdot \hat{\mathbf{a}} = 0$ and as in the last part of Theorem 4.8, we can show that $\hat{\mathbf{d}} \times \hat{\mathbf{a}} = -(\hat{\mathbf{d}} - \hat{\mathbf{a}})$. No other solution exists as there is no other light-like vector orthogonal to $\hat{\mathbf{d}} - \hat{\mathbf{a}}$ other than $\hat{\mathbf{d}} - \hat{\mathbf{a}}$ itself. □

Proof of 3. Finally, if $\hat{\mathbf{d}} - \hat{\mathbf{a}}$ is time-like, no solution exists since if we again inspect Equation (4.1.25), the left hand side is space-like or light-like while the right hand side is time-like. □

We conclude this section by mentioning the trivial case.

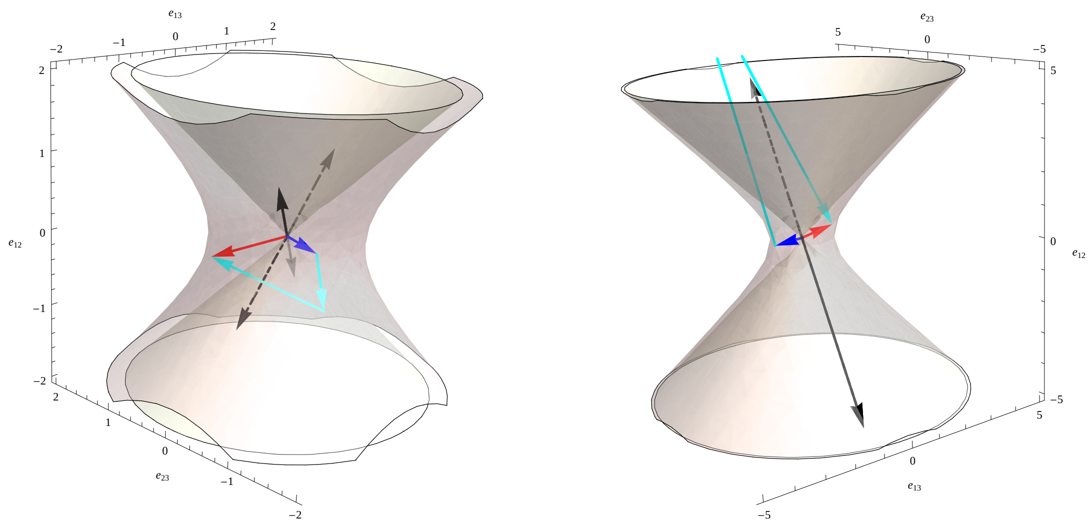
Theorem 4.11. *The multivectors $X = \pm 1$ satisfy Equation* (4.0.1) *if and only if* $\mathbf{a} = \mathbf{d}$. □

4.2 Space-like solutions

Unlike in the previous case with time-like solutions, the unique description of a negative unit multivector from $\mathcal{C\ell}^+_{2,1}$ given in Equation (2.5.20) reduces the number of cases needed to investigate. Furthermore, we will find that with the computations from Theorems 4.5 through 4.10, our work is largely already done. Firstly, substitute $X = \left[\sinh\frac{\psi}{2}, \cosh\frac{\psi}{2}\hat{\mathbf{x}}_-\right]$ in Equation (4.0.1). Expanding the left hand side yields

$$\begin{aligned}\widehat{X}\hat{\mathbf{a}}\widehat{X}^\dagger = &-\cosh^2\frac{\psi}{2}(\hat{\mathbf{x}}_- \cdot \hat{\mathbf{a}})\hat{\mathbf{x}}_- + \sinh^2\frac{\psi}{2}\hat{\mathbf{a}} + \\ &2\cosh\frac{\psi}{2}\sinh\frac{\psi}{2}(\hat{\mathbf{x}}_- \times \hat{\mathbf{a}}) - \cosh^2\frac{\psi}{2}\big((\hat{\mathbf{x}}_- \times \hat{\mathbf{a}}) \times \hat{\mathbf{x}}_-\big) = \hat{\mathbf{d}}.\end{aligned} \tag{4.2.1}$$

Figure 4.6: An example of usage of Theorem 4.10 with vectors $\hat{\mathbf{a}}$ in blue and $\hat{\mathbf{d}}$ in red. The set of bivector parts of all solutions are in dashed black.



(a) $\hat{\mathbf{a}} \times \hat{\mathbf{d}}$ is not light-like.

(b) $\hat{\mathbf{a}} \times \hat{\mathbf{d}}$ is light-like.

By Theorem 2.30, we can write $\hat{\mathbf{a}}$ as $(\hat{\mathbf{x}}_- \cdot \hat{\mathbf{a}})\hat{\mathbf{x}}_- - (\hat{\mathbf{x}}_- \times \hat{\mathbf{a}}) \times \hat{\mathbf{x}}_-$ and $\hat{\mathbf{d}}$ as $(\hat{\mathbf{x}}_- \cdot \hat{\mathbf{d}})\hat{\mathbf{x}}_- - (\hat{\mathbf{x}}_- \times \hat{\mathbf{d}}) \times \hat{\mathbf{x}}_-$. This, along with applying known hyperbolometric identities simplifies Equation (4.2.1) to

$$\begin{aligned} \widehat{X}\hat{\mathbf{a}}\widehat{X}^\dagger &= -(\hat{\mathbf{x}}_- \cdot \hat{\mathbf{a}})\hat{\mathbf{x}}_- + \sinh\psi(\hat{\mathbf{x}}_- \times \hat{\mathbf{a}}) - \cosh\psi\big((\hat{\mathbf{x}}_- \times \hat{\mathbf{a}}) \times \hat{\mathbf{x}}_-\big) \\ &= (\hat{\mathbf{x}}_- \cdot \hat{\mathbf{d}})\hat{\mathbf{x}}_- - (\hat{\mathbf{x}}_- \times \hat{\mathbf{d}}) \times \hat{\mathbf{x}}_-. \end{aligned} \tag{4.2.2}$$

Comparing Equations (4.1.10) and (4.2.2), we see that the space-like solution reverses the component of $\hat{\mathbf{a}}$ parallel to $\hat{\mathbf{x}}_-$ but performs the same rotation. By comparing the terms of (4.2.2) parallel to $\hat{\mathbf{x}}_-$, we can see that

$$\hat{\mathbf{x}}_- \cdot (\hat{\mathbf{a}} + \hat{\mathbf{d}}) = 0. \tag{4.2.3}$$

In words, $\hat{\mathbf{x}}_-$ must be orthogonal to vector $\hat{\mathbf{a}} + \hat{\mathbf{d}}$. Once such vectors are found, we need to verify that each of them also satisfies

$$\sinh\psi(\hat{\mathbf{x}}_- \times \hat{\mathbf{a}}) - \cosh\psi\big((\hat{\mathbf{x}}_- \times \hat{\mathbf{a}}) \times \hat{\mathbf{x}}_-\big) = -(\hat{\mathbf{x}}_- \times \hat{\mathbf{d}}) \times \hat{\mathbf{x}}_-. \tag{4.2.4}$$

From here, we begin discussing separate cases.

Theorem 4.12. *If* $\hat{\mathbf{a}}$ *and* $\hat{\mathbf{d}}$ *are both time-like, Equation* (4.0.3) *has a 1-parameter family of space-like solutions, whose bivector parts form:*

1. *An ellipse if* $\hat{\mathbf{a}}$ *and* $\hat{\mathbf{d}}$ *are within the same light cone.*
2. *A hyperbola if* $\hat{\mathbf{a}}$ *and* $\hat{\mathbf{d}}$ *are not within the same light cone.*

Proof of 1. If $\hat{\mathbf{a}}$ and $\hat{\mathbf{d}}$ are time-like and within the same light-cone, $\hat{\mathbf{a}}+\hat{\mathbf{d}}$ is also time-like and hence all negative unit vectors orthogonal to $\hat{\mathbf{a}}+\hat{\mathbf{d}}$ are space-like and by Theorem 2.29 form an ellipse which can be described by

$$\hat{\mathbf{x}}_{-}(\theta) = \cos\theta\frac{(\hat{\mathbf{d}}\times\hat{\mathbf{a}})}{\sqrt{|\mathrm{N}(\hat{\mathbf{d}}\times\hat{\mathbf{a}})|}} + \sin\theta\frac{(\hat{\mathbf{d}}-\hat{\mathbf{a}})}{\sqrt{|\mathrm{N}(\hat{\mathbf{d}}-\hat{\mathbf{a}})|}}, \qquad \theta\in(-\pi,\pi]. \tag{4.2.5}$$

□

Proof of 2. If $\hat{\mathbf{a}}$ and $\hat{\mathbf{d}}$ are not in the same light-cone, $\hat{\mathbf{a}}+\hat{\mathbf{d}}$ is space-like and hence all negative unit vectors orthogonal to $\hat{\mathbf{a}}+\hat{\mathbf{d}}$ form a hyperbola which can be described by

$$\hat{\mathbf{x}}_{-}(\theta) = \cosh\theta\frac{(\hat{\mathbf{d}}\times\hat{\mathbf{a}})}{\sqrt{|\mathrm{N}(\hat{\mathbf{d}}\times\hat{\mathbf{a}})|}} + \sinh\theta\frac{(\hat{\mathbf{d}}-\hat{\mathbf{a}})}{\sqrt{\mathrm{N}(\hat{\mathbf{d}}-\hat{\mathbf{a}})}}, \qquad \theta\in\mathbb{R}. \tag{4.2.6}$$

The rest of the proof is similar to Theorem 4.7. □

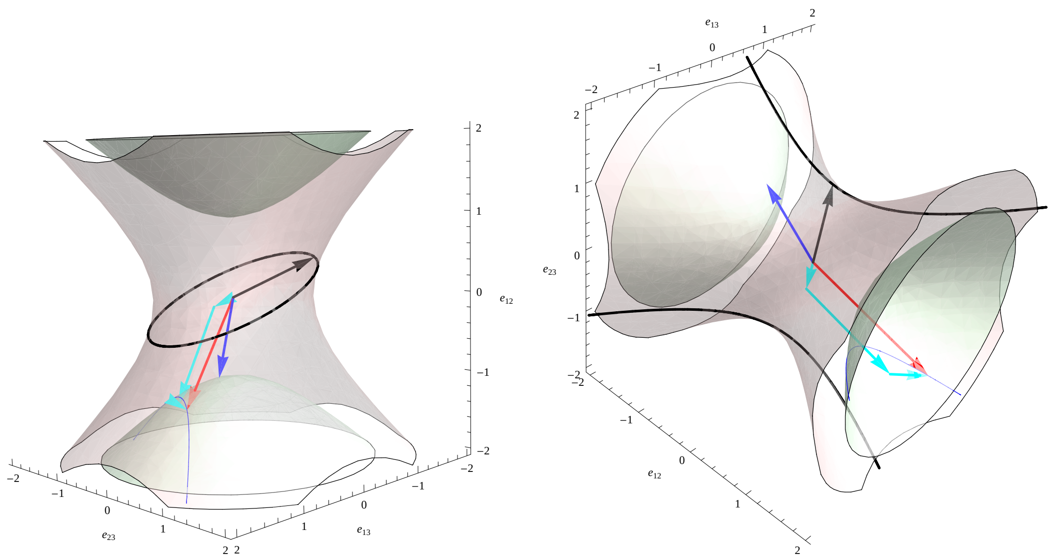
Theorem 4.13. *If* $\hat{\mathbf{a}}$ *and* $\hat{\mathbf{d}}$ *are both space-like, Equation* (4.0.3) *has a 1-parameter family of space-like solutions. In particular:*

1. *A hyperbola with four or two points excluded if* $\hat{\mathbf{d}}+\hat{\mathbf{a}}$ *is space-like.*
2. *An ellipse with four points excluded if* $\hat{\mathbf{d}}+\hat{\mathbf{a}}$ *is time-like.*
3. *A pair of parallel lines with two points excluded if* $\hat{\mathbf{d}}+\hat{\mathbf{a}}$ *is light-like.*

Proof of 1. If $\hat{\mathbf{a}}$, $\hat{\mathbf{d}}$ and $\hat{\mathbf{d}}+\hat{\mathbf{a}}$ are space-like and $\hat{\mathbf{d}}\times\hat{\mathbf{a}}$ is not light-like, then all negative unit vectors orthogonal to $\hat{\mathbf{a}}+\hat{\mathbf{d}}$ form a hyperbola which can be described by

$$\hat{\mathbf{x}}_{-+}(\theta) = \cosh\theta\frac{(\hat{\mathbf{d}}-\hat{\mathbf{a}})}{\sqrt{|\mathrm{N}(\hat{\mathbf{d}}-\hat{\mathbf{a}})|}} + \sinh\theta\frac{(\hat{\mathbf{d}}\times\hat{\mathbf{a}})}{\sqrt{\mathrm{N}(\hat{\mathbf{d}}\times\hat{\mathbf{a}})}}, \qquad \theta\in\mathbb{R}, \tag{4.2.7}$$

Figure 4.7: An example of usage of Theorem 4.12 with vectors $\hat{\mathbf{a}}$ in blue and $\hat{\mathbf{d}}$ in red. The set of bivector parts of all solutions is the thick black curve. The thin blue curve is formed by the results of the rotation for a fixed $\hat{\mathbf{x}}_-$ and varying hyperbolic angle ψ.



(a) $\hat{\mathbf{a}}$ and $\hat{\mathbf{d}}$ are within the same light cone.   (b) $\hat{\mathbf{a}}$ and $\hat{\mathbf{d}}$ are not within the same light cone.

if $\hat{\mathbf{d}} \times \hat{\mathbf{a}}$ is time-like. The four excluded vectors have parameter values θ satisfying

$$\tanh^2\theta = \left|\frac{\mathrm{N}(\hat{\mathbf{d}} \times \hat{\mathbf{a}})}{\hat{\mathbf{d}} - \hat{\mathbf{a}}}\right|. \tag{4.2.8}$$

If $\hat{\mathbf{d}} \times \hat{\mathbf{a}}$ is space-like, the hyperbola can be described by

$$\hat{\mathbf{x}}_{--}(\theta) = \cosh\theta \frac{(\hat{\mathbf{d}} \times \hat{\mathbf{a}})}{\sqrt{|\mathrm{N}(\hat{\mathbf{d}} \times \hat{\mathbf{a}})|}} + \sinh\theta \frac{(\hat{\mathbf{d}} - \hat{\mathbf{a}})}{\sqrt{\mathrm{N}(\hat{\mathbf{d}} - \hat{\mathbf{a}})}}, \qquad \theta \in \mathbb{R}, \tag{4.2.9}$$

with the vectors excluded satisfying

$$\coth^2\theta = \left|\frac{\mathrm{N}(\hat{\mathbf{d}} \times \hat{\mathbf{a}})}{\hat{\mathbf{d}} - \hat{\mathbf{a}}}\right|. \tag{4.2.10}$$

If $\hat{\mathbf{d}} \times \hat{\mathbf{a}}$ is light-like, we again construct the hyperbola of orthogonal negative unit vectors $\hat{\mathbf{x}}_{-0}(\theta)$ using the method from Theorem 4.6. There are two exceptions, with parameter θ satisfying

$$\mathrm{N}(\hat{\mathbf{x}}_{-0}(\theta) \times \hat{\mathbf{a}}) = 0. \tag{4.2.11}$$

The remainder of the proof is similar to Theorem 4.8, Part 1. $\square$

Proof of 2. If $\hat{\mathbf{a}}+\hat{\mathbf{d}}$ is time-like, then all negative unit vectors orthogonal to $\hat{\mathbf{a}}+\hat{\mathbf{d}}$ form an ellipse which can be described by

$$\hat{\mathbf{x}}_-(\theta) = \cos\theta\frac{(\hat{\mathbf{d}}\times\hat{\mathbf{a}})}{\sqrt{|\mathrm{N}(\hat{\mathbf{d}}\times\hat{\mathbf{a}})|}} + \sin\theta\frac{(\hat{\mathbf{d}}-\hat{\mathbf{a}})}{\sqrt{|\mathrm{N}(\hat{\mathbf{d}}-\hat{\mathbf{a}})|}}, \qquad \theta\in\mathbb{R}, \tag{4.2.12}$$

with four excluded vectors for θ values satisfying

$$\cot^2\theta = \left|\frac{\mathrm{N}(\hat{\mathbf{d}}\times\hat{\mathbf{a}})}{\mathrm{N}(\hat{\mathbf{d}}-\hat{\mathbf{a}})}\right|. \tag{4.2.13}$$

The remainder of the proof is similar to Theorem 4.8, Part 2. $\square$

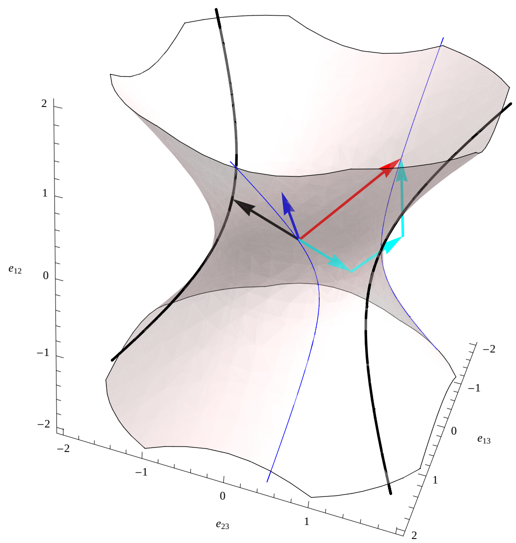
Proof of 3. If $\hat{\mathbf{a}}+\hat{\mathbf{d}}$ is light-like, the plane orthogonal to $\hat{\mathbf{a}}+\hat{\mathbf{d}}$ contains $\hat{\mathbf{a}}+\hat{\mathbf{d}}$ and its intersection with the negative unit hyperboloid forms two lines parallel to $\hat{\mathbf{a}}+\hat{\mathbf{d}}$. Observe that $\hat{\mathbf{a}}$ and $\hat{\mathbf{d}}$ each lie on a separate line. Comparing this to Theorem 4.8, Part 3 and adopting the same notation, we see that this corresponds with reversing the component of $\hat{\mathbf{a}}$ parallel to $\hat{\mathbf{x}}_-$ we observed in Equation (4.2.2). The rest of the computation is identical to Theorem 4.8, Part 3 and shows that for any such $\hat{\mathbf{x}}_-$ except for $\hat{\mathbf{a}}$ and $-\hat{\mathbf{a}}$, there exists a hyperbolic angle ψ such that $\hat{\mathbf{d}} = -\hat{\mathbf{x}}_- \pm e^{\mp}\hat{\mathbf{a}}_\perp$ or $\hat{\mathbf{d}} = -\hat{\mathbf{x}}_- \mp e^{\pm}\hat{\mathbf{a}}_\perp = \hat{\mathbf{d}}$. Once again, ψ cannot however be defined in terms of $\hat{\mathbf{a}}_\perp$ and $\hat{\mathbf{d}}_\perp$. $\square$

4.3 Solutions for light-like a

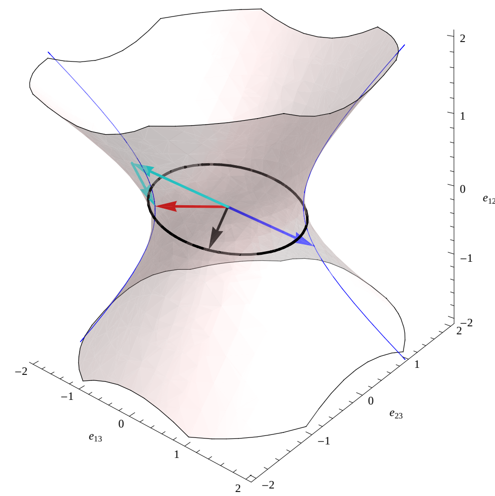
We now drop of the assumption $\mathrm{N}(\mathbf{a})\mathrm{N}(\mathbf{d}) \neq 0$, which allowed us to transform Equation (4.0.1) into the unit form (4.0.3). First, let us consider the case when $\mathrm{N}(\mathbf{a}) = 0$. Because Theorem 4.1 immediately implies that $\mathrm{N}(\mathbf{d}) = 0$, we are therefore looking for all multivectors X which transform one light-like vector into another, without any immediate restrictions on $\mathrm{N}(X)$.

Theorem 4.14. *If* $\mathbf{a}$ *and* $\mathbf{d}$ *are both light-like, Equation* (4.0.1) *has a 1-parameter family of time-like solutions with time-like bivector parts if* $\mathbf{a}$ *and* $\mathbf{d}$ *lie on the same light cone (future or past).*

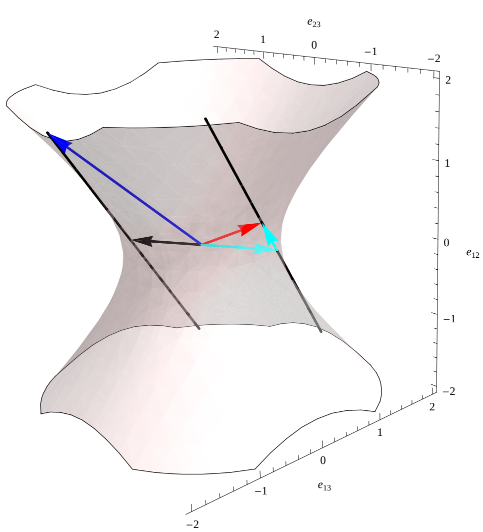
Figure 4.8: An example of usage of Theorem 4.13 with vectors $\hat{\mathbf{a}}$ in blue and $\hat{\mathbf{d}}$ in red. The set of bivector parts of all solutions is the thick black curve. The thin blue curve is formed by the results of the rotation for a fixed $\hat{\mathbf{x}}_-$ and varying hyperbolic angle ψ.



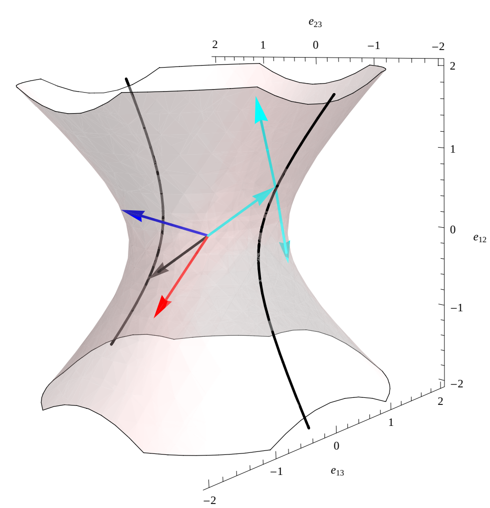
(a) $\hat{\mathbf{d}} + \hat{\mathbf{a}}$ is space-like.



(b) $\hat{\mathbf{d}} + \hat{\mathbf{a}}$ is time-like.



(c) $\hat{\mathbf{d}} + \hat{\mathbf{a}}$ is light-like.



(d) The singular case; $\hat{\mathbf{x}}_- \times \hat{\mathbf{a}}$ is parallel to $(\hat{\mathbf{x}}_- \times \hat{\mathbf{a}}) \times \hat{\mathbf{x}}_-$.

Proof. By Equation (2.5.19), we can write any time-like multivector X with a time-like bivector part as $\sqrt{\mathrm{N}(X)}(\cos\frac{\phi}{2} + \sin\frac{\phi}{2}\hat{\mathbf{x}}_+)$ where $\mathrm{N}(\hat{\mathbf{x}}_+) = 1$ and $\phi \in [-\pi, \pi)$. Substituting into (4.0.1) and expanding yields after simplification

$$\begin{aligned} X\mathbf{a}X^\dagger &= \mathrm{N}^2(X)\Big(-(\hat{\mathbf{x}}_+ \cdot \mathbf{a})\,\hat{\mathbf{x}}_+ + \sin\phi\,(\hat{\mathbf{x}}_+ \times \mathbf{a}) + \cos\phi\big((\hat{\mathbf{x}}_+ \times \mathbf{a}) \times \hat{\mathbf{x}}_+\big)\Big) \\ &= -(\hat{\mathbf{x}}_+ \cdot \mathbf{d})\hat{\mathbf{x}}_+ + (\hat{\mathbf{x}}_+ \times \hat{\mathbf{d}}) \times \hat{\mathbf{x}}_+. \end{aligned} \tag{4.3.1}$$

By comparing the components parallel to $\hat{\mathbf{x}}_+$, we immediately see that this forces $N(X) = 1$ and $\mathbf{a}$ with $\mathbf{d}$ being on the same light cone. Furthermore, $\hat{\mathbf{x}}_+$ must be orthogonal to the vector $\mathbf{d} - \mathbf{a}$ which is space-like. If we now recall Lemma 4.4 for a time-like and a light-like vector, the rest of the proof is identical to Theorem 4.5, with the hyperbola of bivector parts of the solution described by

$$\hat{\mathbf{x}}_+(\theta) = \pm\cosh\theta\frac{(\mathbf{a} + \mathbf{d})}{\sqrt{\mathrm{N}(\mathbf{a} + \mathbf{d})}} + \sinh\theta\frac{(\mathbf{d} \times \mathbf{a})}{\sqrt{|\mathrm{N}(\mathbf{d} \times \mathbf{a})|}}, \qquad \theta \in \mathbb{R}. \tag{4.3.2}$$

□

Theorem 4.15. *If* $\mathbf{a}$ *and* $\mathbf{d}$ *are both light-like, Equation* (4.0.3) *has a 1-parameter family of time-like solutions with space-like bivector parts with two points excluded. In particular:*

1. *A hyperbola if* $\mathbf{a}$ *and* $\mathbf{d}$ *lie on the same light cone.*
2. *An ellipse if* $\mathbf{a}$ *and* $\mathbf{d}$ *do not lie on the same light cone but* $\mathbf{a} \times \mathbf{d} \neq 0$.
3. *A pair of parallel lines if* $\mathbf{a} \times \mathbf{d} = 0$.

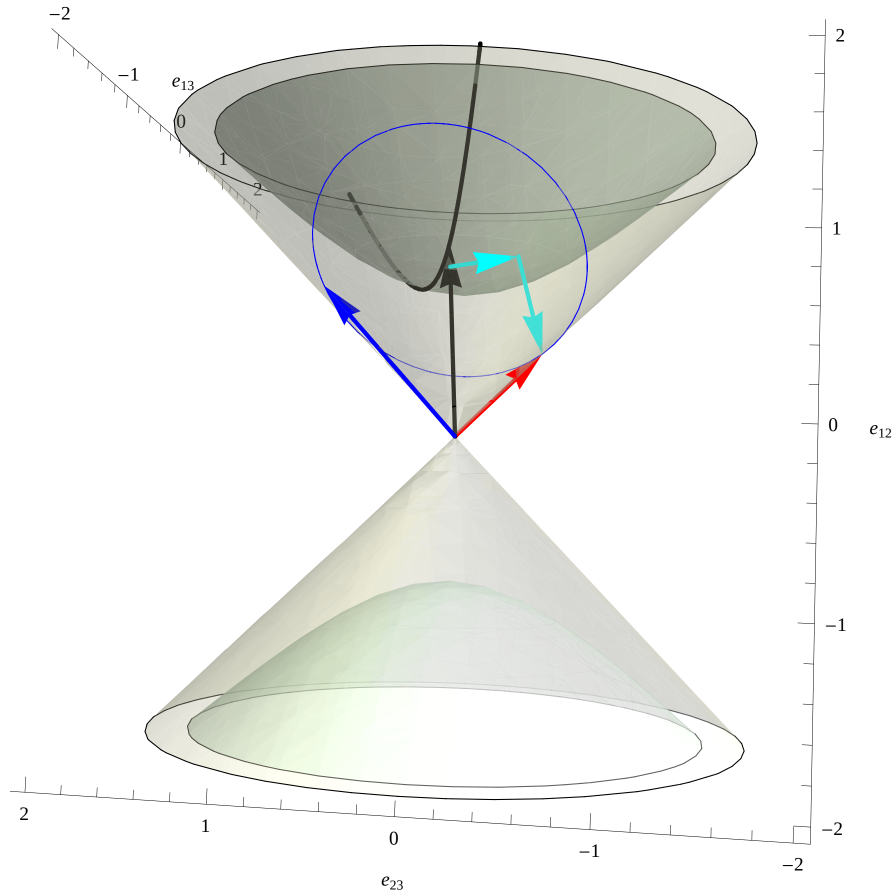
Proof of 1. By Equation (2.5.19), we can write any time-like multivector X with a space-like bivector part as $\sqrt{\mathrm{N}(X)}(\cosh\psi + \sinh\psi\hat{\mathbf{x}}_-)$ where $\mathrm{N}(\hat{\mathbf{x}}_-) = -1$ and $\psi \in \mathbb{R}$. Substituting into (4.0.1) and expanding yields after simplification

$$\begin{aligned} X\mathbf{a}X^\dagger &= \mathrm{N}^2(X)\Big((\hat{\mathbf{x}}_- \cdot \mathbf{a})\,\hat{\mathbf{x}}_- + \sinh\psi\,(\hat{\mathbf{x}}_- \times \mathbf{a}) - \cosh\psi\big((\hat{\mathbf{x}}_- \times \mathbf{a}) \times \hat{\mathbf{x}}_-\big)\Big) \\ &= -(\hat{\mathbf{x}}_- \cdot \mathbf{d})\hat{\mathbf{x}}_- + (\hat{\mathbf{x}}_- \times \hat{\mathbf{d}}) \times \hat{\mathbf{x}}_-. \end{aligned} \tag{4.3.3}$$

This again forces $N(X) = 1$. Furthermore, $\hat{\mathbf{x}}_-$ must be orthogonal to the vector $\mathbf{d} - \mathbf{a}$. This gives us the hyperbola

$$\hat{\mathbf{x}}_-(\theta) = \cosh\theta\frac{(\mathbf{d} \times \mathbf{a})}{\sqrt{|\mathrm{N}(\mathbf{d} \times \mathbf{a})|}} + \sinh\theta\frac{(\mathbf{a} + \mathbf{d})}{\sqrt{\mathrm{N}(\mathbf{a} + \mathbf{d})}}, \qquad \theta \in \mathbb{R}, \tag{4.3.4}$$

Figure 4.9: An example of usage of Theorem 4.14 with vectors $\hat{\mathbf{a}}$ in blue and $\hat{\mathbf{d}}$ in red. The set of bivector parts of all solutions is the thick black hyperbola. The thin blue ellipse is the set of results of the rotation for a fixed $\hat{\mathbf{x}}_+$ and varying circular angle ϕ.



if $\mathbf{a}$ and $\mathbf{d}$ lie on the same light cone. Inspecting the expression

$$\mathrm{N}(\hat{\mathbf{x}}_- \times \mathbf{a}) = \cosh^2\theta \frac{\mathrm{N}\big((\mathbf{d}\times\mathbf{a})\times\mathbf{a}\big)}{|\mathrm{N}(\mathbf{d}\times\mathbf{a})|} + \sinh^2\theta \frac{\mathrm{N}(\mathbf{d}\times\mathbf{a})}{|\mathrm{N}(\mathbf{d}+\mathbf{a})|}, \tag{4.3.5}$$

we can see that $\mathrm{N}\big((\mathbf{d}\times\mathbf{a})\times\mathbf{a}\big)$ is always 0, hence $\mathrm{N}(\hat{\mathbf{x}}_-\times\mathbf{a}) = 0$ if $\sinh\psi = 0$, which occurs exactly twice, once on each branch of the hyperbola. □

Proof of 2. If $\mathbf{a}$ and $\mathbf{d}$ do not lie on the same light cone, the negative unit vectors orthogonal to $\mathbf{d}-\mathbf{a}$ form an ellipse described by

$$\hat{\mathbf{x}}_-(\theta) = \cos\theta \frac{(\mathbf{a}+\mathbf{d})}{\sqrt{|\mathrm{N}(\mathbf{a}+\mathbf{d})|}} + \sin\theta \frac{(\mathbf{d}\times\mathbf{a})}{\sqrt{|\mathrm{N}(\mathbf{d}\times\mathbf{a})|}}, \qquad \theta \in (-\pi, \pi], \tag{4.3.6}$$

unless $\mathbf{a} = -\mathbf{d}$. Again, we can see that in the expression

$$\mathrm{N}(\hat{\mathbf{x}}_- \times \mathbf{a}) = \cos^2\theta \frac{\mathrm{N}(\mathbf{d}\times\mathbf{a})}{|\mathrm{N}(\mathbf{d}+\mathbf{a})|} + \sin^2\theta \frac{\mathrm{N}\big((\mathbf{d}\times\mathbf{a})\times\mathbf{a}\big)}{|\mathrm{N}(\mathbf{d}\times\mathbf{a})|}, \tag{4.3.7}$$

the term $\mathrm{N}\big((\mathbf{d}\times\mathbf{a})\times\mathbf{a}\big)$ always equals zero, and therefore the whole expression is zero if $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$. The rest of the proof is identical to Theorem 4.7. $\square$

Proof of 3. If $\mathbf{a}\times\mathbf{d} = 0$, then $\mathbf{a} = -\alpha\mathbf{d}$ for some $\alpha > 0$. The plane orthogonal to $\mathbf{d}-\mathbf{a}$ contains $\mathbf{d}-\mathbf{a}$ and its intersection with the negative unit hyperboloid forms two parallel lines. Since both $\mathbf{a}$ and $\mathbf{d}$ are now orthogonal to $\hat{\mathbf{x}}_-$, Equation (4.3.3) simplifies to

$$X\mathbf{a}X^\dagger = \sinh\psi(\hat{\mathbf{x}}_- \times \mathbf{a}) + \cosh\psi\mathbf{a} = \mathbf{d}. \tag{4.3.8}$$

As in Theorem 4.8, Part 3, using spherical coordinates we can show that $(\hat{\mathbf{x}} \times \mathbf{a}) = \pm\mathbf{a}_\perp = \pm\mathbf{a}$. Recalling that we allow cosh to be negative and applying known hyperbolometric identities we obtain

$$X\mathbf{a}X^\dagger = -e^{\pm\psi}\mathbf{a}. \tag{4.3.9}$$

Therefore we can always find a hyperbolic angle ψ such that $-e^{\pm\psi}\mathbf{a} = \mathbf{d}$. $\square$

Theorem 4.16. *If* $\mathbf{a}$ *and* $\mathbf{d}$ *are both light-like, Equation* (4.0.1) *has two 1-parameter families of time-like solutions with light-like bivector parts forming two parabolas with two points excluded if* $\mathbf{a}$ *and* $\mathbf{d}$ *lie on the same light-cone and no solutions otherwise.*

Proof. By Equation (2.5.19), we can write any time-like multivector with a light-like bivector part as $X(n) = n[1,\mathbf{x}]$ for some $n \neq 0$. Recalling Theorem 4.9, we can expand Equation (4.0.1) to

$$X\mathbf{a}X^\dagger = n^2\big(\mathbf{a} - 2(\mathbf{x}\cdot\mathbf{a})\mathbf{x} \pm 2(\mathbf{x}\times\mathbf{a})\big) = \mathbf{d}. \tag{4.3.10}$$

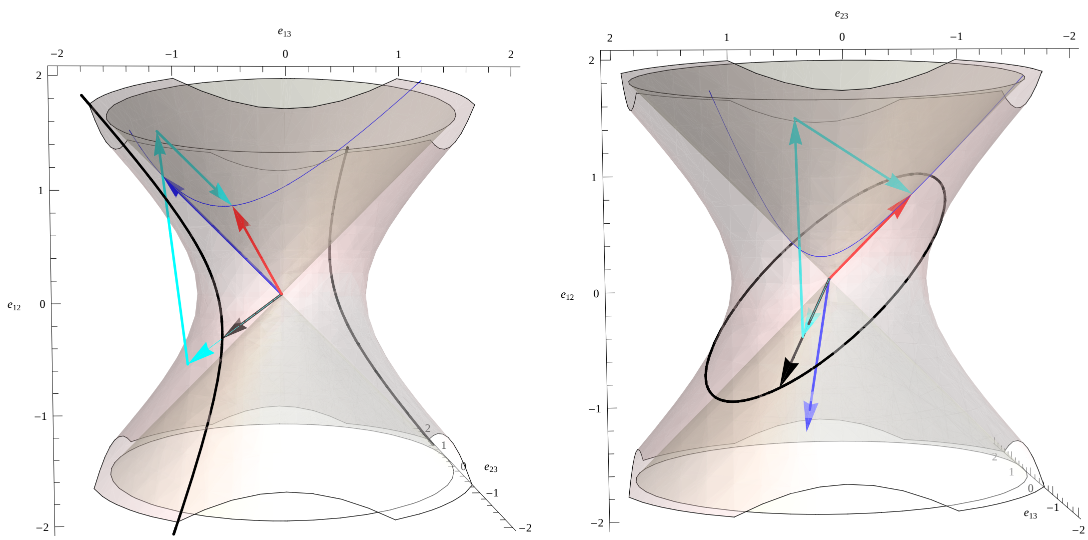
This forces

$$-2(\mathbf{x}\cdot\mathbf{a})\mathbf{x} \pm 2(\mathbf{x}\times\mathbf{a}) = \frac{\mathbf{d}-n^2\mathbf{a}}{n^2}. \tag{4.3.11}$$

Observe that both vectors on the left hand side of Equation (4.3.11) are orthogonal to $\mathbf{x}$, hence $\mathbf{d}-n^2\mathbf{a}$ must also be orthogonal to $\mathbf{x}$. If we now compare the Clifford norms of both sides, we can see that

$$4\mathrm{N}(\mathbf{x}\times\mathbf{a}) = \frac{\mathrm{N}(\mathbf{d}-n^2\mathbf{a})}{n^4},$$

Figure 4.10: An example of usage of Theorem 4.15 with vectors $\hat{\mathbf{a}}$ in blue and $\hat{\mathbf{d}}$ in red. The set of bivector parts of all solutions is the thick black curve. The thin blue curve is formed by the results of the rotation for a fixed $\hat{\mathbf{x}}_-$ and varying hyperbolic angle ψ.



(a) $\hat{\mathbf{a}}$ and $\hat{\mathbf{d}}$ are on the same light cone.

(b) $\hat{\mathbf{a}}$ and $\hat{\mathbf{d}}$ are not on the same light cone.

and therefore $\mathbf{a}$ and $\mathbf{d}$ must lie on the same light cone since $\mathbf{x} \times \mathbf{a}$ must be space-like.

It is easy to see that any light-like vector orthogonal to $\mathbf{d} - \mathbf{a}$ must be a multiple of

$$\mathbf{x}_{0\pm}(n) = \frac{\mathbf{d} + n^2\mathbf{a}}{\sqrt{\mathrm{N}(\mathbf{d} + n^2\mathbf{a})}} \pm \frac{\mathbf{d} \times \mathbf{a}}{\sqrt{|\mathrm{N}(\mathbf{d} \times \mathbf{a})}}. \tag{4.3.12}$$

We will examine the structure of these vectors later; for now, we need to find the scaling coefficient. Substituting $\alpha\mathbf{x}_0$ for some $\alpha \in \mathbb{R}$ into Equation (4.3.11) and comparing the Clifford norms yields

$$4\alpha^2 \frac{\mathrm{N}(\mathbf{d} \times \mathbf{a})}{\mathrm{N}(\mathbf{d} + n^2\mathbf{a})} = \frac{\mathrm{N}(\mathbf{d} - n^2\mathbf{a})}{n^4} \quad \Rightarrow \quad \alpha = \pm 1. \tag{4.3.13}$$

We now instead of assuming X to be of the form $n[1, \alpha\mathbf{x}]$ choose the representation $[n, \alpha n\mathbf{x}]$. Scaling the Equation (4.3.12) by αn gives us

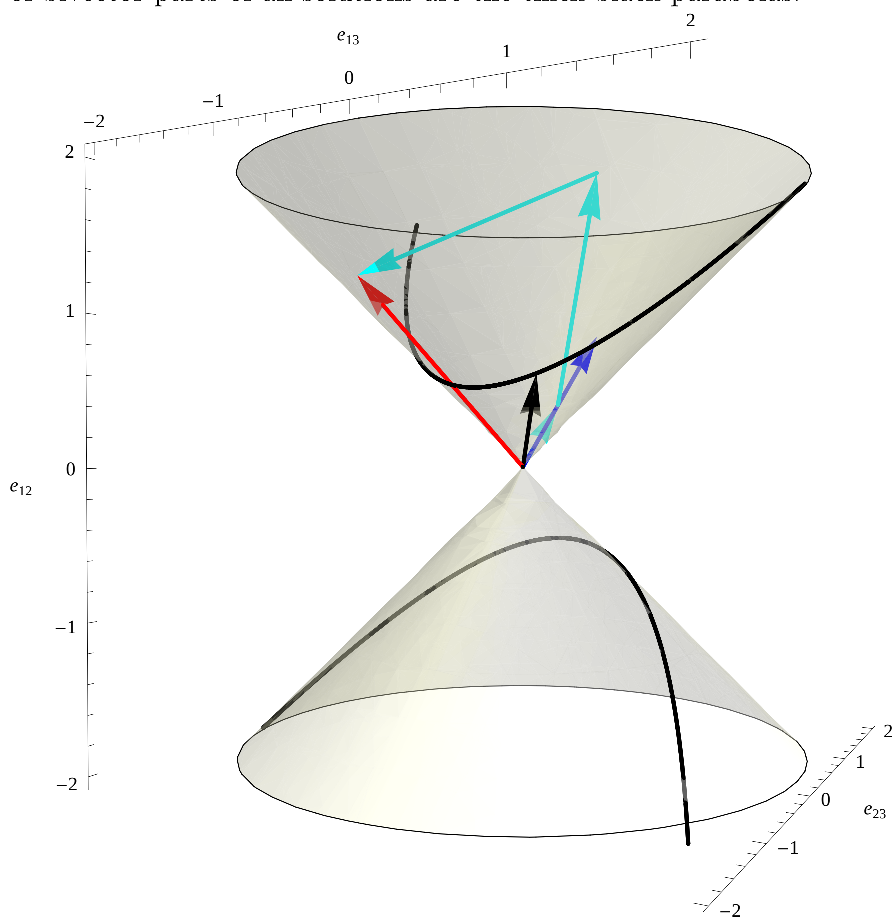
$$\begin{aligned} \alpha n \mathbf{x}_{0\pm}(n) &= \alpha \left(\frac{n\mathbf{d} + n^3\mathbf{a}}{\sqrt{\mathrm{N}(\mathbf{d} + n^2\mathbf{a})}} \pm \frac{n(\mathbf{d} \times \mathbf{a})}{\sqrt{|\mathrm{N}(\mathbf{d} \times \mathbf{a})|}} \right) \\ &= \alpha \left(\frac{\mathbf{d}}{\sqrt{2|\mathbf{a} \cdot \mathbf{d}|}} + \frac{n^2\mathbf{a}}{\sqrt{2|\mathbf{a} \cdot \mathbf{d}|}} \pm \frac{n(\mathbf{d} \times \mathbf{a})}{|\mathbf{a} \cdot \mathbf{d}|} \right). \end{aligned} \tag{4.3.14}$$

In words, $\alpha n\mathbf{x}_{0\pm}(n)$ is a light-like combination of a constant light-like vector $\frac{\mathbf{d}}{\sqrt{2|\mathbf{a}\cdot\mathbf{d}|}}$ with a variable space-like vector $\frac{n^2\mathbf{a}}{\sqrt{2|\mathbf{a}\cdot\mathbf{d}|}} \pm \frac{n(\mathbf{d}\times\mathbf{a})}{|\mathbf{a}\cdot\mathbf{d}|}$ residing in the plane orthogonal to $\mathbf{a}$. Therefore $\alpha n\mathbf{x}_{0\pm}(n)$ forms two parabolas on the intersection of the light cone and the plane orthogonal to $\mathbf{a}$ shifted by $\pm\frac{\mathbf{d}}{\sqrt{2|\mathbf{a}\cdot\mathbf{d}|}}$. Observe that because of the coefficient $\pm n$, the parabolas are doubly covered. However, only half of the vectors correspond to a solution of (4.3.10), namely

$$X(n) = [n, \pm\mathbf{x}_{0\mp}].$$

The two vectors excluded from the solution correspond to the value $n = 0$. $\square$

Figure 4.11: An example of usage of Theorem 4.16 with vectors $\hat{\mathbf{a}}$ in blue and $\hat{\mathbf{d}}$ in red. The set of bivector parts of all solutions are the thick black parabolas.



Theorem 4.17. *If* $\hat{\mathbf{a}}$ *and* $\hat{\mathbf{d}}$ *are both light-like, Equation* (4.0.1) *has a 1-parameter family of space-like solutions with two points excluded. In particular:*

1. *A hyperbola if* $\hat{\mathbf{d}} + \hat{\mathbf{a}}$ *is space-like.*

2. *An ellipse if* $\hat{\mathbf{d}} + \hat{\mathbf{a}}$ *is time-like.*

3. *A pair of parallel lines if* $\hat{\mathbf{d}} \times \hat{\mathbf{a}} = 0$.

Proof of 1. By Equation (2.5.20), we can write any space-like multivector X as $\sqrt{\mathrm{N}(X)}(\sinh\frac{\psi}{2} + \cosh\frac{\psi}{2}\hat{\mathbf{x}}_-)$ where $\mathrm{N}(\hat{\mathbf{x}}_-) = -1$ and $\psi \in \mathbb{R}$. Substituting into (4.0.1) and expanding yields after simplification

$$\begin{aligned} \widehat{X}\mathbf{a}\widehat{X}^\dagger &= \mathrm{N}^2(X)\Big((\hat{\mathbf{x}}_- \cdot \mathbf{a})\hat{\mathbf{x}}_- + \sinh\psi(\hat{\mathbf{x}}_- \times \mathbf{a}) - \cosh\psi\big((\hat{\mathbf{x}}_- \times \mathbf{a}) \times \hat{\mathbf{x}}_-\big)\Big) \\ &= (\hat{\mathbf{x}}_- \cdot \mathbf{d})\hat{\mathbf{x}}_- - (\hat{\mathbf{x}}_- \times \mathbf{d}) \times \hat{\mathbf{x}}_-. \end{aligned} \tag{4.3.15}$$

By comparing the components parallel to $\hat{\mathbf{x}}_-$, we see that $\mathrm{N}(X)$ must equal -1. The rest of the proof is similar to Theorem 4.15.

If $\mathbf{a} + \mathbf{d}$ is space-like, the set of negative unit vectors orthogonal to $\mathbf{a} + \mathbf{d}$ forms a hyperbola with equation

$$\hat{\mathbf{x}}_-(\theta) = \cosh\theta\frac{(\mathbf{d} \times \mathbf{a})}{\sqrt{|\mathrm{N}(\mathbf{d} \times \mathbf{a})|}} + \sinh\theta\frac{(\mathbf{d} - \mathbf{a})}{\sqrt{|\mathrm{N}(\mathbf{d} - \mathbf{a})|}}, \qquad \theta \in \mathbb{R}. \tag{4.3.16}$$

Inspecting the expression

$$\mathrm{N}(\hat{\mathbf{x}}_- \times \mathbf{a}) = \cosh^2\theta\frac{\mathrm{N}\big((\mathbf{d} \times \mathbf{a}) \times \mathbf{a}\big)}{|\mathrm{N}(\mathbf{d} \times \mathbf{a})|} + \sinh^2\theta\frac{\mathrm{N}(\mathbf{d} \times \mathbf{a})}{|\mathrm{N}(\mathbf{d} - \mathbf{a})|}, \tag{4.3.17}$$

we see that $\mathrm{N}\big((\mathbf{d} \times \mathbf{a}) \times \mathbf{a}\big)$ always vanishes. Thus the vectors $\hat{\mathbf{x}}_-(\theta)$ excluded from the solution satisfy $\sinh\theta = 0$. □

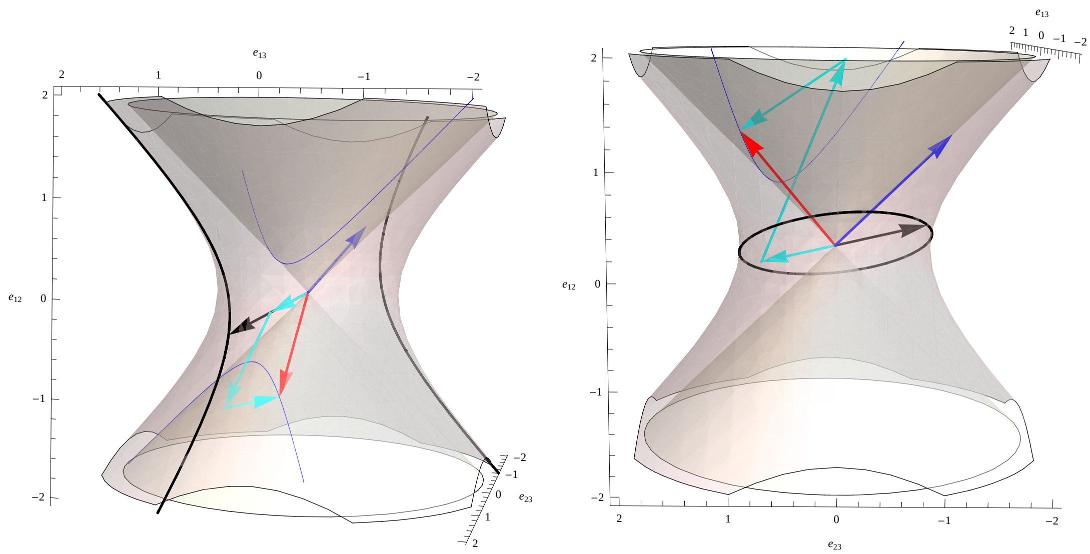
Proof of 2. If $\mathbf{a} + \mathbf{d}$ is time-like, the set of negative unit vectors orthogonal to $\mathbf{a} + \mathbf{d}$ forms an ellipse with equation

$$\hat{\mathbf{x}}_-(\theta) = \cos\theta\frac{(\mathbf{d} \times \mathbf{a})}{\sqrt{|\mathrm{N}(\mathbf{d} \times \mathbf{a})|}} + \sin\theta\frac{(\mathbf{d} - \mathbf{a})}{\sqrt{|\mathrm{N}(\mathbf{d} - \mathbf{a})|}}, \qquad \theta \in (-\pi, \pi]. \tag{4.3.18}$$

The pair of vectors excluded from the solution satisfy $\theta = \pm\pi$. □

Proof of 3. This case is identical to Theorem 4.15 Part 3 since the vector $\mathbf{a}$ has no part parallel to $\hat{\mathbf{x}}_-$. □

Figure 4.12: An example of usage of Theorem 4.17 with vectors $\hat{\mathbf{a}}$ in blue and $\hat{\mathbf{d}}$ in red. The set of bivector parts of all solutions is the thick black curve. The thin blue curve is formed by the results of the rotation for a fixed $\hat{\mathbf{x}}_-$ and varying hyperbolic angle ψ.



(a) $\hat{\mathbf{d}} + \hat{\mathbf{a}}$ is space-like.

(b) $\hat{\mathbf{d}} + \hat{\mathbf{a}}$ is time-like.

4.4 Solutions for light-like d

The last case requiring an analysis is the case when $\mathrm{N}(\mathbf{d}) = 0$ and $\mathrm{N}(\mathbf{a}) \neq 0$. From Theorem 4.1, we see that this implies $\mathrm{N}(X) = 0$ with no restrictions on $\mathrm{N}(\mathbf{a})$. Equation (2.5.21) then gives us two possibilities.

Theorem 4.18. *If $X = [0, \alpha\mathbf{x}_0]$ is a light-like multivector with light-like bivector part $\mathbf{x}_0$ and $\alpha \neq 0$, Equation (4.0.1) has exactly one solution $X = \frac{\mathbf{d}}{\sqrt{|\mathbf{a}\cdot\mathbf{d}|}}$ or $X = -\frac{\mathbf{d}}{\sqrt{|\mathbf{a}\cdot\mathbf{d}|}}$, if $\mathbf{a}\cdot\mathbf{d} \neq 0$, and no solutions otherwise.*

Proof. Substituting into Equation (4.0.1) and expanding yields

$$X\mathbf{a}X^\dagger = -\alpha^2(\mathbf{x}_0 \times \mathbf{a}) \times \mathbf{x}_0. \tag{4.4.1}$$

If we now investigate the formula for $(\mathbf{x}_0 \times \mathbf{a}) \times \mathbf{x}_0$ given in the proof of Theorem (2.30), we will find that if $\mathbf{x}_0$ is light-like, the expression $(\mathbf{x}_0 \times \mathbf{a}) \times \mathbf{x}_0$ can be transformed into $(\mathbf{x}_0 \cdot \mathbf{a})\mathbf{x}_0$. We therefore set either $\mathbf{x}_0 = \mathbf{d}$ or $\mathbf{x}_0 = -\mathbf{d}$ based on the sign of $\mathbf{d}\cdot\mathbf{a}$ so that Equation (4.4.1) is satisfied. $\square$

We were unfortunately unable to reach a rigorous proof on the existence and structure of solutions for the last remaining case where $X = \alpha[\pm 1, \hat{\mathbf{x}}_-]$ for some negative unit vector $\hat{\mathbf{x}}_-$ and $\alpha \neq 0$. Applying substitutions

$$\widehat{X} = \frac{\sqrt{2}X}{\alpha\sqrt[4]{|\mathbf{a}|}}, \qquad \hat{\mathbf{a}} = \frac{\mathbf{a}}{\sqrt{|\mathrm{N}(\mathbf{a})|}},$$

allows us to transform Equation (4.0.1) into

$$\widehat{X}\hat{\mathbf{a}}\widehat{X}^\dagger = \frac{1}{2}\Big(- (\hat{\mathbf{x}}_- \cdot \hat{\mathbf{a}})\hat{\mathbf{x}}_- + \hat{\mathbf{a}} + 2\hat{\mathbf{x}}_- \times \hat{\mathbf{a}} - (\hat{\mathbf{x}}_- \times \hat{\mathbf{a}}) \times \hat{\mathbf{x}}_-\Big) = \mathbf{d},$$

which after decomposing $\hat{\mathbf{a}}$ and $\mathbf{d}$ by Theorem 2.30 simplifies to

$$\hat{\mathbf{x}}_- \times \hat{\mathbf{a}} - (\hat{\mathbf{x}}_- \times \hat{\mathbf{a}}) \times \hat{\mathbf{x}}_- = (\hat{\mathbf{x}}_- \cdot \mathbf{d})\hat{\mathbf{x}}_- - (\hat{\mathbf{x}}_- \times \mathbf{d}) \times \hat{\mathbf{x}}_-.$$

Since entire left hand side is now orthogonal to $\hat{\mathbf{x}}_-$, we see that $\hat{\mathbf{x}}_- \cdot \mathbf{d} = 0$, therefore $\hat{\mathbf{x}}_-$ is coplanar with $\mathbf{d}$ and lies on one of the parallel lines of negative unit vectors. Furthermore, this implies $(\hat{\mathbf{x}}_- \times \mathbf{d}) \times \hat{\mathbf{x}}_- = \mathbf{d}$. The Clifford norm of the left hand side is clearly zero by Lemma 4.4. We were unsuccessful in searching for further results, however, numerical experiments suggest that solutions exist for each $\hat{\mathbf{x}}_-$ from one of the two parallel lines. One of the possible ways for reaching the proof might be using spherical coordinates to describe vectors $\hat{\mathbf{x}}_-$, $\hat{\mathbf{a}}$ and $\mathbf{d}$ and assembling a system of equations for the three coordinates.

4.5 Summary

In Theorems 4.6, 4.7 and 4.12, we assumed the bivector part of X or the given vectors $\mathbf{a}$ and $\mathbf{d}$ to be time-like, the solutions formed a continuous family of rotations around unit axes orthogonal to the vector $\mathbf{d} - \mathbf{a}$. These cases correspond perfectly with the solutions to equation $U\mathbf{v}U^* = \mathbf{w}$ in $\mathbb{H}$.

If both the bivector part of X and vectors $\mathbf{a}$ and $\mathbf{d}$ were space-like such as in Theorems 4.8 and 4.13, while the solutions still maintained the orthogonal property, we identified up to four point cases when the obtained vectors did not provide a solution to the equation. We also noted that the structure of solutions transitions between the cases in an expected manner, with the degenerate case of two parallel lines occuring at the breaking point between elliptical and hyperbolical solution.

For time-like solutions with light-like bivector parts, we obtained a finite number of solutions, however the orthogonality to $\mathbf{d} - \mathbf{a}$ is preserved.

The space-like solutions are generally similar to time-like solutions with space-like bivector parts, but with a different orthogonal property.

The solutions to the case of transforming light-like vectors were largely analogous to transforming time-like vectors, except for the unique case of a time-like solution with light-like bivector part (Theorem 4.16), which formed a pair of parabolas.

Apart from a trivial case, we were unsuccessful in our search for a light-like solution.

Chapter 5

Conclusion and Discussion

Our goal for this paper was to study selected quadratic equations, in particular Clifford algebras $\mathcal{C}\ell_{1,1}^{+}$ and $\mathcal{C}\ell_{2,1}^{+}$, which arise from the study of Minkowski Pythagorean hodograph curves. We were interested in such equations which arise from the related topic of Pythagorean hodograph curves and whose solutions are already known for the quaternions. We conclude that the solution procedures used in the quaternion cases can be easily approved in the Clifford algebra setting and provide the expected number of solutions with similar structure.

We would like to emphasize the singular cases found in Theorems 4.8 and 4.13. These phenomena were not mentioned in any of the studies of linear transformations in Minkowski space $\mathbb{R}^{2,1}$ that were available to us.

In future studies, we would like to verify our conjecture about existence of light-like solutions to the equation $X\mathbf{a}X^{\dagger} = \mathbf{d}$ and use the completed classification of solutions in solving the equation $XAX^{\dagger} + BX^{\dagger} + XC + D = 0$ in $\mathcal{C}\ell_{2,1}^{+}$, which was part of our motivation for this research.

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